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Hilbert's Double Series Theorem and Principal Latent Roots of the Resulting Matrix

The inequality $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n-1} \leqq \pi \sum_p a_p^2$, was proved by HILBERT and published by WEYL.[1] Various proofs were given by HARDY, LITTLEWOOD & PÓLYA.[2] In this inequality π is the best possible constant; that is, the maximum value of $\sum_m\sum_n \frac{a_m a_n}{m+n-1}\Big/\sum_p a_p^2$ for arbitrary $\{a_p\}$ is π. It is no longer the best possible sum when the summation is finite; from 1 to N, say. In this case FRAZER[3] has shown that $(N+1)\sin[\pi/(N+1)]$ is better. But this result is not the best possible, and COPSEY, FRAZER, & SAWYER have published investigations[4] based on empirical values of the constant λ for $N = 1(1)5,10,20$, computed by the Royal Aircraft Establishment. Further computations for $N = 2(1)20$ are being made by the National Physical Laboratory.

The ordinary method for maximizing this quadratic form shows[5] that the best possible value of the constant is the greatest latent root of the matrix $\|1/(m+n-1)\|$; $n \leqq N$, and it was in this way that the values were computed.

The roots and vectors of the segments of the Hilbert matrices of N rows were derived from the iterated multiplication of the matrix into an arbitrary column vector, the procedure and accelerating processes being essentially those of AITKEN.[6] Operations were carried out on British Hollerith machines following the cycle, tabulator-reproducer-sorter-multiplier-tabulator. Machines were checked in the usual way by check sums. Final checking was done on a Brunsviga 20 machine, and roots and vectors for $n = 6$ and 8 were added at this stage.

An approximate relation between N and λ of the form $1/(\pi - \lambda) = a \ln(N + b) + c$ has been found,[4] but it is evidently incomplete as the error increases rapidly with N. Dr. OLGA TODD, in a recent unpublished communication, has shown that though λ tends to π asymptotically as N tends to ∞, π is not a latent root of the infinite matrix. Further, she shows that $\lambda = \pi\{1 + O(1/\log N)\}$ and $\sum_1^N\sum_1^N \frac{a_m a_n}{m+n-1}\Big/\sum_1^N a_p^2 \sim \pi\{1 + O(1/\log N)\}$, when $\{a_p\} = \{1, 1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{4}, \cdots 1/\sqrt{N}\}$.

Principal Latent Root and Vector of Segments of the Hilbert Matrix

N	2	3	4	10	20
Latent Root	1.26759 188	1.40831 893	1.50021 428	1.75191 967	1.90713 472
Latent Vector	1	1	1	1	1
	0.53518 376	0.55603 256	0.57017 208	0.60899 191	0.63153 893
		0.39090 795	0.40677 899	0.45313 830	0.48170 552
			0.31814 097	0.36528 601	0.39577 939
				0.30775 305	0.33864 052
				0.26672 518	0.29732 839
				0.23580 131	0.26579 806
				0.21156 396	0.24080 108
				0.19200 513	0.22041 627
				0.17586 003	0.20342 569
					0.18901 536

N	5	6	8	20 *concl.*
Latent Root	1.56705 069	1.61889 986	1.69593 900	0.17661 823
				0.16582 577
				0.15633 540
				0.14791 772
Latent Vector	1	1	1	0.14039 536
	0.58056 692	0.58862 854	0.60050 425	0.13362 876
	0.41880 095	0.42832 928	0.44267 155	0.12750 652
	0.33006 105	0.33966 189	0.35437 045	0.12193 851
	0.27325 824	0.28252 359	0.29691 858	0.11685 095
		0.24233 781	0.25618 093	
			0.22562 937	
			0.20179 019	

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[1] H. WEYL, *Singuläre Integralgleichungen mit besonderer Berücksichtigung des Fourierschen Integraltheorems*. Diss. Göttingen, 1908.

[2] G. H. HARDY, J. E. LITTLEWOOD, & G. PÓLYA, *Inequalities*, Cambridge, 1934, p. 226–259.

[3] H. FRAZER, "Note on Hilbert's inequality," London Math. Soc., *Jn.*, v. 21, 1946, p. 7–9.

[4] E. H. COPSEY, H. FRAZER, & W. W. SAWYER, "Empirical data on Hilbert's inequality," *Nature*, v. 161, 6 Mar. 1948, p. 361.

[5] R. COURANT & D. HILBERT, *Methoden der mathem. Physik*, second ed., v. 1, Berlin, 1931; U.S.A. photo-lithoprint, 1943.

[6] A. C. AITKEN, "Studies in practical mathematics. II. The evaluation of the latent roots and latent vectors of a matrix," R. Soc. Edinb., *Proc.*, v. 57, p. 269–304, 1937.

Piecewise Polynomial Approximation for Large-Scale Digital Calculators

1. Introduction. Most large-scale digital calculating machines are equipped to perform automatically the arithmetic operations of addition, subtraction, multiplication, division, and in some cases of extracting the square root. All arithmetic processes must be carried out by suitably combining these given operations. But many functions whose evaluation is frequently required, such as the elementary transcendental functions, for example, cannot be represented exactly by any combination of a finite number of the given operations. In order to evaluate such functions, it is necessary to resort to some sort of approximation.

A method frequently employed may be called "piecewise polynomial approximation." This method consists of dividing the interval upon which the required function is to be approximated into a number of sub-intervals upon each of which the function is represented by a polynomial. The coefficients of these polynomials are stored within the machine or external to it in a manner consistent with the machine's construction. When the value of the independent variable is given, the proper sub-range is selected by the machine itself. The operations of addition and multiplication applied to the value of the independent variable and to the stored coefficients are then sufficient to evaluate the appropriate polynomial and hence to obtain an approximation to the required function.

In practice, the range over which the approximation is to hold and the maximum allowable error are usually known in advance. We shall assume that the maximum allowable degree of the approximating polynomials is

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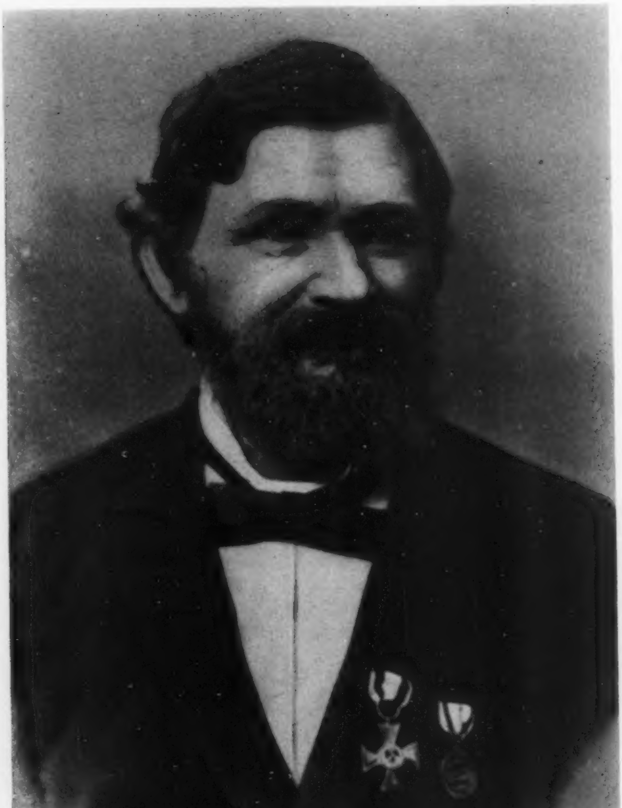

D. F. E. MEISSEL 1890

given. The problem of piecewise polynomial approximation reduces, then, to the determination of the sub-intervals and the coefficients of approximating polynomials so as to be consistent with these specified quantities. This may be stated more precisely as follows:

PROBLEM—Given the function $f(x)$ defined on the interval $[\alpha, \beta]$, a specified constant positive tolerance T, and a specified positive integer N. Required to divide $[\alpha, \beta]$ into sub-intervals $[c_{i-1}, c_i]$ where, with the number of sub-intervals, r, as yet unspecified, $i = 1,2,\cdots r$, and either $\alpha = c_0 < c_1 < \cdots < c_{r-1} < c_r = \beta$, or $\beta = c_0 > c_1 > \cdots > c_{r-1} > c_r = \alpha$, and to determine n^{th} degree polynomials $P_n{}^i(x)$ with $n \leq N$, such that the upper bound of $|f(x) - P_n{}^i(x)| \leq T$ on $[c_{i-1}, c_i]$.

If the quantities c_i and the polynomials $P_n{}^i(x)$ are determined in such a way that the number of sub-intervals, r, shall be a minimum, then they will be said to constitute the best solution of the problem. We suppose that $f(x)$ is a continuous function, possessing as many continuous derivatives as shall be required, and that all of these derivatives shall have a finite number of zeros.

2. Determination of Sub-Intervals. We shall restrict ourselves to approximation by n^{th} degree polynomials agreeing with $f(x)$ at $n + 1$ points on $[c_{i-1}, c_i]$. With the $n + 1$ points of coincidence specified, say $x = x_k{}^i$ $(k = 0,1,\cdots,n)$, any such polynomial $P_n{}^i(x)$ may be expressed by the Lagrange Interpolation Formula,

$$P_n{}^i(x) = \sum_{k=0}^{n} \frac{Q_{n+1}^i(x)f(x_k{}^i)}{(x - x_k{}^i)Q_{n+1}^{i(1)}(x_k{}^i)}, \tag{1}$$

where, $Q_{n+1}^i(x) = (x - x_0{}^i)(x - x_1{}^i)\cdots(x - x_n{}^i)$, and $Q_{n+1}^{i(1)}(x)$ denotes the derivative of $Q_{n+1}^i(x)$. The remainder term, $f(x) - P_n{}^i(x)$, is then given by

$$R_{n+1}^i(x) = Q_{n+1}^i(x)f^{(n+1)}(\xi_i)/(n + 1)!$$

where ξ_i lies on $[c_{i-1}, c_i]$.

Suppose that $f^{(n+1)}(x)\cdot f^{(n+2)}(x) \leq 0$ on $[\alpha, \beta]$. In this case, designate the end-points of the sub-intervals by $c_0, c_1, \cdots, c_r$ in order of increasing subscripts from left to right. The upper bound of $|f^{(n+1)}(x)|$ on $[c_{i-1}, c_i]$ occurs at $x = c_{i-1}$. Denote by $Q_{n+1}^{i\,\max}$ the upper bound of $|Q_{n+1}^i(x)|$ on $[c_{i-1}, c_i]$. Then

$$|R_{n+1}^i(x)| \leq |Q_{n+1}^{i\,\max}f^{(n+1)}(c_{i-1})|/(n + 1)!. \tag{2}$$

Let us transform the independent variable in such a way that the interval $[c_{i-1}, c_i]$ becomes $[-1,1]$.

$$x = \tfrac{1}{2}(c_i - c_{i-1})u + \tfrac{1}{2}(c_i + c_{i-1}); \qquad u = (2x - c_i - c_{i-1})/(c_i - c_{i-1}). \tag{3}$$

Denote the transform of $Q_{n+1}^i(x)$ by $[\tfrac{1}{2}(c_i - c_{i-1})]^{n+1}L_{n+1}(u)$. Since $Q_{n+1}^i(x)$ is of leading coefficient unity, so is $L_{n+1}(u)$. In fact,

$$L_{n+1}(u) = (u - u_0)(u - u_1)\cdots(u - u_n),$$

where $u_0, u_1, \cdots, u_n$ are the points into which $x_0{}^i, x_1{}^i, \cdots, x_n{}^i$, respectively, are transformed, and u ranges on the interval $[-1,1]$. Denote by $L_{n+1}^{\max}$ the upper bound of $|L_{n+1}(u)|$ on $[-1,1]$. Now

$$Q_{n+1}^{i\,\max} = [\tfrac{1}{2}(c_i - c_{i-1})]^{n+1}L_{n+1}^{\max},$$

and therefore from (2),

$$|R^i_{n+1}(x)| \leq |[\tfrac{1}{2}(c_i - c_{i-1})]^{n+1} L^{\max}_{n+1} f^{(n+1)}(c_{i-1})|/(n+1)!.$$

We wish to determine the division points $c_i (i = 0, 1, \cdots, r)$ in such a way that

$$|R^i_{n+1}(x)| \leq T \quad \text{on} \quad [c_{i-1}, c_i].$$

This condition will surely be satisfied if

$$(4) \qquad |[\tfrac{1}{2}(c_i - c_{i-1})]^{n+1} L^{\max}_{n+1} f^{(n+1)}(c_{i-1})|/(n+1)! \leq T \quad \text{on} \quad [c_{i-1}, c_i].$$

Solving (4) for c_i, we obtain

$$(5) \qquad c_i \leq c_{i-1} + 2|(n+1)!T/[L^{\max}_{n+1} f^{(n+1)}(c_{i-1})]|^{\frac{1}{n+1}},$$

a condition which may be used to generate successive end-points from left to right. If the equality in expression (5) holds, $[c_{i-1}, c_i]$ will be called a complete sub-interval. If the inequality holds, it will be called an incomplete sub-interval. Note that it is in general impossible to derive from (4) a condition for generating the end-points from right to left, since c_{i-1} does not enter algebraically in this expression.

If $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \geq 0$ on $[\alpha, \beta]$, we designate the end-points of the sub-intervals by $c_r, c_{r-1}, \cdots, c_0$ from left to right. An inequality analogous to (5) may be derived. In either case, the condition which successive end-points must satisfy suggests a procedure for the determination of the sub-intervals. This procedure may be stated as follows:

RULE: Generate the quantities c_i by the recurrence formula

$$(6) \qquad c_i = c_{i-1} \pm K/|f^{(n+1)}(c_{i-1})|^{\frac{1}{n+1}}, \text{ where}$$

$$(7) \qquad K = 2\{[(n+1)!T]/L^{\max}_{n+1}\}^{\frac{1}{n+1}}.$$

If $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \leq 0$ on $[\alpha, \beta]$, start with $c_0 = \alpha$, use the plus sign in (6), and continue the recurrence process until some quantity, say c_s, greater than or equal to β is obtained. c_{r-1} is then taken to be c_{s-1} and c_r taken to be β. If $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \geq 0$ on $[\alpha, \beta]$, start with $c_0 = \beta$, use the minus sign in (6) and continue the recurrence process until some quantity, say c_s, less than or equal to α is obtained. c_{r-1} is then taken to be c_{s-1} and c_r taken to be α.

If neither $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \leq 0$ nor $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \geq 0$ over the entire range $[\alpha, \beta]$, we may divide $[\alpha, \beta]$ into sub-ranges upon which these conditions will hold alternately. This is always possible. We may then apply the foregoing rule to each sub-range separately, taking for c_0 one of the end-points of the sub-range in question. This procedure will result in several incomplete sub-intervals of the type $[c_{r-1}, c_r]$ being employed upon the range $[\alpha, \beta]$. All but one of the incomplete sub-intervals could be eliminated by choosing the c_0's in a less naive manner, but the saving achieved seems hardly worth the additional complication.

3. A First Order Approximation to the Number of Sub-Intervals Required. Let $h_i = c_i - c_{i-1}$. We have from (6),

$$(8) \qquad |h_i| = K/|f^{(n+1)}(c_{i-1})|^{\frac{1}{n+1}}.$$

When $f^{(n+1)}(x)\cdot f^{(n+2)}(x) \leq 0$, $|f^{(n+1)}(x)|$ decreases with increasing x, and hence, since the sub-intervals are generated from left to right, $|f^{(n+1)}(c_{i-1})|$ decreases with increasing i. When $f^{(n+1)}(x)\cdot f^{(n+2)}(x) \geq 0$, $|f^{(n+1)}(x)|$ increases with increasing x. But since the sub-intervals are in this case generated from right to left, $|f^{(n+1)}(c_{i-1})|$ again decreases with increasing i. In either case, the following theorem follows directly from (8).

THEOREM I. Over an interval in which the sign of $f^{(n+1)}(x)\cdot f^{(n+2)}(x)$ does not change, the length of each complete sub-interval generated is greater than or equal to the length of the immediately preceding sub-interval.

Also from (8)

THEOREM II. When $f(x)$ is an $(n + 1)^{\text{th}}$ degree polynomial, all complete sub-intervals are of equal length.

Again, for the case in which $f^{(n+1)}(x)\cdot f^{(n+2)}(x)$ does not alternate in sign throughout the interval $[\alpha, \beta]$, let

$$h_{\min} = K/|f^{(n+1)}(c_0)|^{\frac{1}{n+1}}, \text{ and} \tag{9}$$

$$h_{\max} = K/|f^{(n+1)}(c_r - h_1)|^{\frac{1}{n+1}}. \tag{10}$$

From Theorem I, it follows that

$$h_{\min} \leq |h_i| \leq h_{\max}, \tag{11}$$

where $|h_i|$ is the length of any complete sub-interval. Now

$$|c_j - c_0| = \sum_{i=1}^{j} |h_i|,$$

and hence from (11)

$$jh_{\min} \leq |c_j - c_0| \leq jh_{\max}.$$

Replacing j by r, the number of sub-intervals required to cover the entire range $[\alpha, \beta]$, and recalling that $|c_r - c_0| = \beta - \alpha$, we have

$$(\beta - \alpha)/h_{\max} \leq r \leq 1 + (\beta - \alpha)/h_{\min}, \tag{12}$$

where the quantity, 1, on the right-hand side of (12) enters by virtue of the fact that r must be an integer. We formulate our results as follows:

THEOREM III. The number of sub-intervals, r, required to represent $f(x)$ on $[\alpha, \beta]$ is bounded by the quantities $(\beta - \alpha)/h_{\max}$ and $1 + (\beta - \alpha)/h_{\min}$ where $h_{\min}$ and $h_{\max}$ are given by (9) and (10), respectively.

4. A Second Order Approximation to the Number of Sub-Intervals Required. Expressions for determining the lengths of the sub-intervals in terms of $f^{(n+2)}(\xi)$ can also be derived, but due to the indefiniteness of the quantity ξ, they are not appreciably more accurate than those developed in the last section, and hence are of little practical value in estimating the number of sub-intervals required. They are, however, of some theoretical interest.

Solving (8) for $|f^{(n+1)}(c_{i-1})|$ and subtracting from the resulting expression a similar expression for $|f^{(n+1)}(c_{i-2})|$, we obtain

$$|f^{(n+1)}(c_{i-1})| - |f^{(n+1)}(c_{i-2})| = K^{n+1}[|h_i|^{-(n+1)} - |h_{i-1}|^{-(n+1)}]. \tag{13}$$

We may, by use of the law of the mean, write

$$(14) \qquad |f^{(n+1)}(c_{i-1})| - |f^{(n+1)}(c_{i-2})| = - |h_{i-1}f^{(n+2)}(\xi_{i-1})|,$$

where ξ_{i-1} lies on $[c_{i-2}, c_{i-1}]$. Substituting for the left-hand side of (13) its value as given by (14), and solving for h_i, we obtain

$$(15) \qquad |h_i| = |h_{i-1}| \{1/[1 - K^{-(n+1)} | h_{i-1}^{n+2} f^{(n+2)}(\xi_{i-1})|]\}^{1/(n+1)}.$$

Theorem II is an immediate consequence of this expression. Equation (15) may be written in the form

$$(16) \qquad K^{-(n+1)} | h_{i-1}^{n+2} f^{(n+2)}(\xi_{i-1})| = 1 - |h_{i-1}/h_i|^{n+1}.$$

By Theorem I, the quantity $|h_{i-1}/h_i|^{n+1}$ is less than or equal to unity. Hence the quantity on the right-hand side of (16) is less than unity and greater than or equal to zero. If $|f^{(n+2)}(\xi_{i-1})|$ increases with increasing i, the quantity in braces in (15) increases with increasing i. We may therefore state

THEOREM IV. *If $f^{(n+1)}(x)\cdot f^{(n+3)}(x) \le 0$, the ratio of the length of any complete sub-interval to the length of the previous one increases with each sub-interval generated.*

The converse of this theorem is not, in general, true.

Let

$$f_{\min}^{(n+2)} = \begin{cases} |f^{(n+2)}(c_0)| & \text{when} \quad f^{(n+1)}(x)\cdot f^{(n+3)}(x) \leqslant 0 \\ |f^{(n+2)}(c_r - h_{\min})| & \text{when} \quad f^{(n+1)}(x)\cdot f^{(n+3)}(x) \geqslant 0 \end{cases}$$

and

$$(17) \qquad M_{\min} = (h_{\min})^{n+2} f_{\min}^{(n+2)}/K^{n+1}.$$

From (15), it follows that

$$|h_i| \ge |h_{i-1}| \{1/(1 - M_{\min})\}^{\frac{1}{n+1}}$$

and, by recurrence

$$|h_i| \ge |h_0| \{1/(1 - M_{\min})\}^{\frac{i-1}{n+1}} \ge h_{\min} + \frac{i-1}{n+1} M_{\min} h_{\min}.$$

Summing from $i = 1$ to j,

$$|c_j - c_0| \ge j h_{\min} + \tfrac{1}{2} j(j-1)\, M_{\min} h_{\min}/(n+1).$$

Replacing j by r and $|c_j - c_0|$ by $\beta - \alpha$, we have

THEOREM V. *The number of sub-intervals, r, required to represent $f(x)$ on $[\alpha, \beta]$ must satisfy the inequality*

$$(18) \qquad \beta - \alpha \ge r h_{\min} + \tfrac{1}{2} r(r-1)\, M_{\min} h_{\min}/(n+1),$$

where $M_{\min}$ is given by (17). Since r enters quadratically in (18), an upper bound to the number of sub-intervals required can easily be determined. For $f(x)$ an $(n+1)^{st}$ degree polynomial, the second term on the right of (18) vanishes.

5. Approximation by Particular Types of Polynomials. If, in $\underset{n+1}{L}(u)$ we let $u_0 = u_1 = \cdots = u_n = 0$, we obtain

$$L_{n+1}(u) = u^{n+1}, \qquad \text{and} \qquad (19) \quad L_{n+1}^{\max} = 1.$$

In this case, the polynomials $P_n{}^i(x)$ given by (1) assume indeterminate forms. The indetermination may be resolved by rearranging terms, setting

$$x_0{}^i = (c_i + c_{i-1})/2,\ x_1{}^i = x_0{}^i + \epsilon,\ x_2{}^i = x_0{}^i + 2\epsilon, \text{ etc.},$$

and passing to the limit.[1] For a given i, $P_n{}^i(x)$ reduces then to the n^{th} degree polynomial consisting of the first $n + 1$ terms of the Taylor's Series expansion about the point $(c_i + c_{i-1})/2$.

If we take $u_k = \cos\left[\frac{1}{2}(2k + 1)/(n + 1)\right]\pi$, $k = 0, 1, \cdots, n$, we obtain $L_{n+1}(u) = T_{n+1}(u)$, and $L_{n+1}^{\max} = 1/2^n$, where $T_{n+1}(u)$ is the Chebyshev Polynomial [2] of the first kind of order $n + 1$, defined by the formula

$$T_0(u) = 1;\quad T_n(u) = 2^{1-n}\cos(n\cdot\cos^{-1} u);\quad n = 1, 2, 3, \cdots.$$

Of all the n^{th} degree polynomials of leading coefficient unity, $T_n(u)$ is known to be the one whose absolute value on the interval $[-1,1]$ has the smallest upper bound.[3] From this property, we may deduce

THEOREM VI. The best of all sets of sub-intervals generated by the fundamental rule is that set obtained by taking $L_{n+1}(u)$ to be the Chebyshev Polynomial of the first kind of order $n + 1$.

But if $f^{(n+1)}(\xi)$ is constant on $[\alpha, \beta]$, the best set of sub-intervals generated by the fundamental rule will be the best of all sets of sub-intervals generated in any manner whatsoever. We thus have

THEOREM VII. For $f(x)$ an $(n + 1)$st degree polynomial, the best solution to the problem of piecewise polynomial approximation is obtained by applying the fundamental rule, taking for $L_{n+1}(u)$ the Chebyshev Polynomial of the first kind of order $n + 1$.

6. Numerical Example. Consider the following numerical example:

EXAMPLE. Required to approximate the function $\sin x$ piecewise by cubic polynomials on the interval $[0, \frac{1}{2}\pi]$ in such a way that $\sin x$ is everywhere on the interval represented to an accuracy of 1×10^{-6}.

We have here $f(x) = \sin x$ (footnote 4), $[\alpha, \beta] = [0, \pi/2]$, $n = 3$, $T = 1 \times 10^{-6}$. Since $f^{(n+1)}(x)\cdot f^{(n+2)}(x) \geq 0$ on $[\alpha, \beta]$, the sub-intervals are to be generated from right to left starting with $c_0 = \frac{1}{2}\pi$.

For the Taylor's Series representation, we have from (7) $K = 2(4! \times 10^{-6})^{\frac{1}{4}} = 0.13998$, and from (6)

$$c_i = c_{i-1} - 0.13998\,(\sin c_{i-1})^{-\frac{1}{4}}. \tag{20}$$

The values of c_i obtained by repeated use of (20) are listed in the second column of Table I. Eleven sub-intervals are required. This is consistent with the bounds given by Theorem III; namely, $r \leq 12.23$; $r \geq 6.85$.

Column 3 of Table I gives values of c_i rounded to two decimals in such a way that $|h_i|$ is always on the small side. For the tabulation of the coefficients, it is convenient to refer each polynomial to the interval $[-1, 1]$. The approximating polynomials are then expressed explicitly as functions of u, where u and x are related by (3). The coefficients of these polynomials are given in the first part of Table II. Table III gives values of each approximating polynomial at the end-points of the sub-interval upon which it is to be used. The remainder should be greatest at these points. As was to be expected, the absolute value of the remainder is in all cases less than 1×10^{-6}.

For the Chebyshev approximation,

$$K = 2(2^3 \times 4! \times 10^{-6})^{\frac{1}{4}} = 0.23541, \quad \text{and} \quad c_i = c_{i-1} - 0.23541 (\sin c_{i-1})^{-\frac{1}{4}}.$$

The unrounded values of c_i are given in Column 4 and the rounded values in Column 5 of Table I. Seven sub-intervals are required. This again is in agreement with values predicted by Theorem III, $r \leq 7.68$; $r \geq 4.63$. Again the approximating polynomials are tabulated as functions of u. Their coefficients are given in the second part of Table II. Table IV gives the value of each approximating polynomial at the end-points and at the mid-point of the sub-interval upon which it is to be used. For the fourth sub-interval, values of $P_3^i(x)$ are also tabulated at the points $u = \cos \frac{1}{4}k\pi (k = 1, 2, 3)$ at which the remainder should be zero, and at the points $u = \cos \frac{1}{8}(2k + 1)\pi$, $k = 0, 1, 2, 3$, at which the absolute value of the remainder should be a maximum. As before, the remainders are all less in absolute value than the prescribed tolerance of 1×10^{-6}.

TABLE I—ENDPOINTS OF SUB-INTERVALS

Sub-Interval	Taylor's Series Unrounded	Taylor's Series Rounded	Chebyshev Polynomials Unrounded	Chebyshev Polynomials Rounded
i	c_i	c_i	c_i	c_i
0	1.5708	1.58	1.5708	1.58
1	1.4308	1.45	1.3354	1.35
2	1.2905	1.31	1.0983	1.12
3	1.1491	1.17	0.8559	0.88
4	1.0059	1.03	0.6034	0.63
5	0.8599	0.89	0.3321	0.36
6	0.7098	0.74	0.0206	0.05
7	0.5541	0.59	0.0000	0.00
8	0.3897	0.43		
9	0.2114	0.26		
10	0.0047	0.06		
11	0.0000	0.00		

TABLE II—COEFFICIENTS OF APPROXIMATING POLYNOMIALS

$$P_3^i(x) = a_0^i + a_1^i u + a_2^i u^2 + a_3^i u^3, \text{ where } u = (2x - c_i - c_{i-1})/(c_i - c_{i-1})$$

Approximation by Taylor's Series

i	$[c_{i-1}, c_i]$		a_0^i	a_1^i	a_2^i	a_3^i
1	1.58	1.45	0.9984 4379	−0.0036 2488	−0.0021 0921	0.0000 0256
2	1.45	1.31	0.9818 5353	−0.0132 7486	−0.0024 0554	0.0000 1084
3	1.31	1.17	0.9457 8400	−0.0227 3574	−0.0023 1717	0.0000 1857
4	1.17	1.03	0.8912 0736	−0.0317 5173	−0.0021 8346	0.0000 2593
5	1.03	0.89	0.8191 9157	−0.0401 4640	−0.0020 0702	0.0000 3279
6	0.89	0.74	0.7277 2560	−0.0514 4013	−0.0020 4673	0.0000 4823
7	0.74	0.59	0.6170 5913	−0.0590 1876	−0.0017 3548	0.0000 5533
8	0.59	0.43	0.4881 7725	−0.0698 1956	−0.0015 6217	0.0000 7447
9	0.43	0.26	0.3381 9668	−0.0799 9141	−0.0012 2174	0.0000 9632
10	0.26	0.06	0.1593 1821	−0.0987 2273	−0.0007 9659	0.0001 6454
11	0.06	0.00	0.0299 9550	−0.0299 8650	−0.0000 1350	0.0000 0450

Approximation by Chebyshev's Polynomials

i	$[c_{i-1}, c_i]$		a_0^i	a_1^i	a_2^i	a_3^i
1	1.58	1.35	0.9944 0787	−0.0121 4388	−0.0065 6828	0.0000 2674
2	1.35	1.12	0.9441 4734	−0.0378 9492	−0.0062 3630	0.0000 8348
3	1.12	0.88	0.8414 7008	−0.0648 3626	−0.0060 5133	0.0001 5551
4	0.88	0.63	0.6852 8780	−0.0910 3392	−0.0053 4686	0.0002 3688
5	0.63	0.36	0.4750 3083	−0.1187 9574	−0.0043 2215	0.0003 6050
6	0.36	0.05	0.2035 6655	−0.1517 5436	−0.0024 4046	0.0006 0692
7	0.05	0.00	0.0249 9740	−0.0249 9221	−0.0000 0781	0.0000 0262

TABLE III—COMPARISON OF TAYLOR'S SERIES APPROXIMATION WITH TRUE VALUE OF SIN x

x	i	u	$P_3{}^i(x)$	true value of sin x	$P_3{}^i(x)$ − sin x
1.58	1	−1	0.9999 5690	0.9999 5765	−0.0000 0075
1.45	1	1	0.9927 1226	0.9927 1299	−0.0000 0073
1.45	2	−1	0.9927 1201	0.9927 1299	−0.0000 0098
1.31	2	1	0.9661 8397	0.9661 8495	−0.0000 0098
1.31	3	−1	0.9661 8400	0.9661 8495	−0.0000 0095
1.17	3	1	0.9207 4966	0.9207 5060	−0.0000 0094
1.17	4	−1	0.9207 4970	0.9207 5060	−0.0000 0090
1.03	4	1	0.8572 9810	0.8572 9899	−0.0000 0089
1.03	5	−1	0.8572 9816	0.8572 9899	−0.0000 0083
0.89	5	1	0.7770 7094	0.7770 7175	−0.0000 0081
0.89	6	−1	0.7770 7077	0.7770 7175	−0.0000 0098
0.74	6	1	0.6742 8697	0.6742 8791	−0.0000 0094
0.74	7	−1	0.6742 8708	0.6742 8791	−0.0000 0083
0.59	7	1	0.5563 6022	0.5563 6102	−0.0000 0080
0.59	8	−1	0.5563 6017	0.5563 6102	−0.0000 0085
0.43	8	1	0.4168 6999	0.4168 7080	−0.0000 0081
0.43	9	−1	0.4168 7003	0.4168 7080	−0.0000 0077
0.26	9	1	0.2570 7985	0.2570 8055	−0.0000 0070
0.26	10	−1	0.2570 7981	0.2570 8055	−0.0000 0074
0.06	10	1	0.0599 6345	0.0599 6401	−0.0000 0056
0.06	11	−1	0.0599 6400	0.0599 6401	−0.0000 0001
0.00	11	1	0.0000 0000	0.0000 0000	0.0000 0000

TABLE IV—COMPARISON OF APPROXIMATION BY CHEBYSHEV'S POLYNOMIALS WITH TRUE VALUE OF SIN x

x	i	u	$P_3{}^i(x)$	true value of sin x	$P_3{}^i(x)$ − sin x
1.58	1	−1.0	0.9999 5673	0.9999 5767	−0.0000 0094
1.465	1	0.0	0.9944 0787	0.9944 0879	−0.0000 0092
1.35	1	1.0	0.9757 2245	0.9757 2336	−0.0000 0091
1.35	2	−1.0	0.9757 2248	0.9757 2336	−0.0000 0088
1.235	2	0.0	0.9441 4734	0.9441 4820	−0.0000 0086
1.12	2	1.0	0.9000 9960	0.9001 0044	−0.0000 0084
1.12	3	−1.0	0.9000 9950	0.9001 0044	−0.0000 0094
1.000	3	0.0	0.8414 7008	0.8414 7098	−0.0000 0090
0.88	3	1.0	0.7707 3800	0.7707 3888	−0.0000 0088
0.88	4	−1.0	0.7707 3798	0.7707 3888	−0.0000 0090
0.8704 8494	4	−0.9238 7953	0.7646 4155	0.7646 4155	0.0000 0000
0.8433 8835	4	−0.7071 0678	0.7469 0132	0.7469 0044	0.0000 0088
0.8028 3543	4	−0.3826 8343	0.7193 2866	0.7193 2867	−0.0000 0001
0.755	4	0.0	0.6852 8780	0.6852 8867	−0.0000 0087
0.7071 6457	4	0.3826 8343	0.6496 8088	0.6496 8087	0.0000 0001
0.6666 1165	4	0.7071 0678	0.6183 2742	0.6183 2656	0.0000 0086
0.6395 1506	4	0.9238 7953	0.5968 0639	0.5968 0640	−0.0000 0001
0.63	4	1.0	0.5891 4390	0.5891 4476	−0.0000 0086
0.63	5	−1.0	0.5891 4392	0.5891 4476	−0.0000 0084
0.495	5	0.0	0.4750 3083	0.4750 3165	−0.0000 0082
0.36	5	1.0	0.3522 7344	0.3522 7423	−0.0000 0079
0.36	6	−1.0	0.3522 7353	0.3522 7423	−0.0000 0070
0.205	6	0.0	0.2035 6655	0.2035 6716	−0.0000 0061
0.05	6	1.0	0.0499 7865	0.0499 7917	−0.0000 0052
0.05	7	−1.0	0.0499 7918	0.0499 7917	0.0000 0001
0.025	7	0.0	0.0249 9740	0.0249 9740	0.0000 0000
0.00	7	1.0	0.0000 0000	0.0000 0000	0.0000 0000

I am indebted to Mrs. HELEN MALONE, of the BRL, for the computation of the numerical example at the end of this paper.

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[1] J. F. STEFFENSEN, *Interpolation*, Baltimore, 1927, p. 22.

[2] For a working list of coefficients and formulae relating to the Chebyshev Polynomials, see C. W. JONES, J. C. P. MILLER, J. F. C. CONN, R. C. PANKHURST, "Tables of Chebyshev polynomials," R. Soc. Edinb., *Proc.*, v. 62A, 1946, p. 187–203. See *MTAC*, v. 2, p. 262.

[3] For proof see STEFAN KACMARZ and HUGO STEINHAUS, *Theorie der Orthogonalreihen*, Warsaw, 1935, p. 111–112.

[4] True values of sin x were obtained from NBSCL, *Tables of Circular and Hyperbolic Sines and Cosines*, New York, 1940.

The Accuracy of Linear Interpolation in Tables of the Mathematics of Finance

Many texts in the Mathematics of Finance give empirical statements with respect to the errors due to linear interpolation in the tables contained in these texts. It is the purpose of this paper to derive some formulae which express the amount of these errors. Formulae for the maximum errors will be obtained when solving by linear interpolation (a) for an unknown time and (b) for an unknown rate in tables of finance. The relative size of the errors in different tables will be considered.

(a) *Unknown time.*

Suppose that we are interpolating in the $(1+i)^n$ table or the $s_{\overline{n}|}$ table for an unknown time, n, and that it has been determined that n falls in the interval (n_1, n_2). Then the time, N as given by linear interpolation[1] is

$$N \equiv n_1 + \frac{(1+i)^n - (1+i)^{n_1}}{(1+i)^{n_2} - (1+i)^{n_1}}.$$

Let $n = n_1 + f$ where $(f < 1)$ and since $n_2 - n_1 = 1$, we have

$$N = n_1 + [(1+i)^f - 1]/i = n_1 + s_{\overline{f}|}.$$

The error, E, due to linear interpolation is

$$(1) \qquad E = n - N = n_1 + f - n_1 - s_{\overline{f}|}, \qquad \text{or} \qquad E = f - s_{\overline{f}|}.$$

Taking the first and second derivatives of E with respect to f, we have

$$dE/df = 1 - (1+i)^f \ln(1+i)/i,$$
$$d^2E/df^2 = -(1+i)^f[\ln(1+i)]^2/i.$$

Since d^2E/df^2 is always negative, we have a maximum error given by solving $dE/df = 0$ for f. This gives

$$(2) \qquad f(\max) = \frac{\ln i - \ln\ln(1+i)}{\ln(1+i)} = \frac{1}{2} + \frac{i}{24} + \cdots.$$

Substituting the value of f given by (2) in (1) gives

$$E(\max) = \frac{\ln i - \ln\ln(1+i)}{\ln(1+i)} + \frac{1}{i} - \frac{1}{\ln(1+i)} = \frac{i}{8} - \frac{i^2}{16} + \cdots.$$

The following theorems may now be presented:

THEOREM I: *The error in finding an unknown time by linear interpolation in the* $(1+i)^n$ *table or the* $s_{\overline{n}|}$ *table is independent of the interval* (n_1, n_2).

THEOREM II: *The maximum error in finding an unknown time by linear interpolation in the* $(1+i)^n$ *table or the* $s_{\overline{n}|}$ *table occurs when* f *is slightly more than halfway through the interval.*

THEOREM III: *The error in finding an unknown time by linear interpolation in the* $(1+i)^n$ *table or the* $s_{\overline{n}|}$ *table is never more than* $\frac{1}{8}$ *of the interest rate per period.*[2] (N is always less than n.)

The error due to linear interpolation in the $v^n = (1+i)^{-n}$ table or the $a_{\overline{n}|}$ table [3] is given as follows:

$$E = n - n_1 - (v^n - v^{n_1})(v^{n_2} - v^{n_1})^{-1}$$
$$= f - (v^f - 1)(v - 1)^{-1} = s_{\overline{1-f}|} - (1 - f).$$

From this result the following theorem may be stated.

THEOREM IV: *The error due to linear interpolation for a given value f in the $(1+i)^n$ table or the $s_{\overline{n}|}$ table is the same for $1-f$ in the v^n table or the $a_{\overline{n}|}$ table (opposite in direction). The maximum error in the two tables is the same.*

(b) *Unknown rate.*

Let us now consider the error due to linear interpolation in solving for an unknown rate. If we are interpolating in the $(1+i)^n$ table and it is observed that i falls between the rates i_1 and i_2 which are given in the table (n a given integer), then the interest rate I, given by linear interpolation, is $I = i_1 + (A - A')(i_2 - i_1)(A'' - A')^{-1}$ where $A = (1+i)^n$, $A' = (1+i_1)^n$, $A'' = (1+i_2)^n$. The error (positive) is

$$E = i - I = i - i_1 - (A - A')(i_2 - i_1)(A'' - A')^{-1}. \tag{3}$$

Taking the first and second derivatives of E with respect to i, we have

$$dE/di = 1 - n(1+i)^{n-1}(i_2 - i_1)(A'' - A')^{-1},$$
$$d^2E/di^2 = -n(n-1)(1+i)^{n-2}(i_2 - i_1)(A'' - A')^{-1}.$$

Since d^2E/di^2 is always negative ($n > 1$), we have a maximum error given by solving $dE/di = 0$ for i. This gives

$$i(\max) = \left[\frac{A''-A'}{n(i_2-i_1)}\right]^{1/(n-1)} - 1 \tag{4}$$
$$= \tfrac{1}{2}(i_1+i_2) + \tfrac{1}{24}(n-2)(i_2-i_1)^2 - \tfrac{1}{24}(n-2)\tfrac{1}{2}(i_1+i_2)(i_2-i_1)^2 + \cdots.$$

Substituting the value of i given by (4) in (3) gives

$$E(\max) = \left[\frac{A''-A'}{n(i_2-i_1)}\right]^{1/(n-1)}\left(1-\frac{1}{n}\right) - (1+i_1) + \frac{A'(i_2-i_1)}{A''-A'},$$

or

$$\begin{aligned} E(\max) \doteq {} & [1+\tfrac{1}{2}(i_1+i_2)+\tfrac{1}{24}(n-2)(i_2-i_1)^2 \\ & -\tfrac{1}{24}(n-2)\tfrac{1}{2}(i_1+i_2)(i_2-i_1)^2]\left(1-\frac{1}{n}\right) \\ & -(1+i_1)+\frac{1}{n}[1+\tfrac{1}{2}(n+1)i_1-\tfrac{1}{2}(n-1)i_2+\tfrac{1}{12}(n^2-1)(i_2-i_1)^2 \\ & -\tfrac{1}{8}(n^2-1)(i_2-i_1)^2(i_1+i_2)] \\ \doteq {} & \tfrac{1}{8}(n-1)(i_2-i_1)^2-(n-1)(7n+4)(48n)^{-1}(i_1+i_2)(i_2-i_1)^2, \end{aligned}$$

or

$$E(\max) < \tfrac{1}{8}(n-1)(i_2-i_1)^2. \tag{5}$$

The following theorems may be presented:

THEOREM V: *The maximum error in solving for an unknown rate in the $(1+i)^n$ table occurs when i is slightly more than the mean of the table rates, i_1, i_2.*

THEOREM VI: *The error* [4] *due to linear interpolation in solving for an unknown rate in the* $(1+i)^n$ *table is never more than* $\frac{1}{8}(n-1)(i_2-i_1)^2$.

We may now state and prove a theorem in regard to the size of the error obtained in the $(1+i)^n$ table and the $s_{\overline{n}|}$ table.

THEOREM VII: *The error obtained in solving for an unknown rate by linear interpolation is always less in the* $s_{\overline{n}|}$ *table than the corresponding error in the* $(1+i)^n$ *table.*

This theorem states that

$$i-i_1-(s-s')(s''-s')^{-1}(i_2-i_1)<i-i_1-(A-A')(A''-A')^{-1}(i_2-i_1)$$

where the errors are positive, and

$$s'=[(1+i_1)^n-1]/i_1, \quad s=[(1+i)^n-1]/i, \quad s''=[(1+i_2)^n-1]/i_2, \quad i_1<i<i_2.$$

This inequality may be reduced to

$$\frac{s-s'}{s''-s'}>\frac{A-A'}{A''-A'}, \tag{6}$$

or

$$\frac{C_2\alpha_1+C_3\alpha_2+C_4\alpha_3\cdots C_n\alpha_{n-1}}{C_2\beta_1+C_3\beta_2+C_4\beta_3\cdots C_n\beta_{n-1}}>\frac{C_1\alpha_1+C_2\alpha_2+C_3\alpha_3\cdots C_n\alpha_n}{C_1\beta_1+C_2\beta_2+C_3\beta_3\cdots C_n\beta_n} \tag{7}$$

where

$$\alpha_K=i^K-i_1^K, \qquad K=1(1)n, \qquad \beta_r=i_2^r-i_1^r, \qquad r=1(1)n,$$

and the C's are the coefficients in the binomial formula. By multiplying the means and extremes of inequality (7), we have

$$(\alpha_1\beta_2-\alpha_2\beta_1)(C_2^2-C_1C_3)+(\alpha_1\beta_3-\alpha_3\beta_1)(C_2C_3-C_1C_4)$$
$$+(\alpha_1\beta_4-\alpha_4\beta_1)(C_2C_4-C_1C_5)+(\alpha_2\beta_3-\alpha_3\beta_2)(C_3^2-C_2C_4)+\cdots>0.$$

All the terms of this expansion may be written in the form,

$$(C_{K+1}C_r-C_KC_{r+1})(\alpha_K\beta_r-\alpha_r\beta_K) \tag{8}$$

where $K<r$ except the last term which is $C_n^2(\alpha_{n-1}\beta_n-\alpha_n\beta_{n-1})$. Now

$$C_{K+1}C_r-C_KC_{r+1}=C_KC_r\frac{(n+1)(r-K)}{(K+1)(r+1)}$$

which is positive. Also, we must consider the sign of

$$\alpha_K\beta_r-\alpha_r\beta_K=(i^K-i_1^K)(i_2^r-i_1^r)-(i^r-i_1^r)(i_2^K-i_1^K),$$

which is positive if

$$\frac{i_2^r-i_1^r}{i_2^K-i_1^K}>\frac{i^r-i_1^r}{i^K-i_1^K} \qquad \text{where} \qquad r>K, \qquad i_2>i>i_1$$

or

$$\frac{(1+\alpha)^r-1}{(1+\alpha)^K-1}>\frac{(1+\beta)^r-1}{(1+\beta)^K-1} \quad \text{where} \quad \alpha=\frac{i_2}{i_1}-1, \quad \beta=\frac{i}{i_1}-1. \tag{9}$$

Expanding inequality (9) by multiplying the means together and the extremes together, we have terms of the form $C_\phi C_{K+\theta}(\alpha^{K+\theta}\beta^\phi - \alpha^\phi\beta^{K+\theta})$ where $\phi = 1, 2, \cdots K, \theta = 1, 2, \cdots (r - K)$, or $C_\phi C_{K+\theta}\alpha^\phi\beta^\phi(\alpha^{K+\theta-\phi} - \beta^{K+\theta-\phi})$ which is positive since $\alpha > \beta$. Hence all the terms (8), of the expansion of (7) are positive so that (6) is true and Theorem VII is proved.

By Theorem VII, *the maximum for the error obtained in* (5) *also applies to the* $s_{\overline{n}|}$ *table.*

If we interpolate for an unknown rate in the v^n table, the error is given by

$$E = i_1 + (V - V')(V'' - V')^{-1}(i_2 - i_1) - i \tag{10}$$

where

$$V = (1 + i)^{-n}, \qquad V' = (1 + i_1)^{-n}, \qquad V'' = (1 + i_2)^{-n}.$$

A maximum error is obtained by solving

$$\frac{dE}{di} = -1 - \frac{n(1 + i)^{n-1}}{V'' - V'}(i_2 - i_1) = 0 \text{ for } i$$

since d^2E/di^2 is always negative. Hence

$$i(\max) = \left[\frac{V' - V''}{n(i_2 - i_1)}\right]^{-1/(n+1)} - 1 = \frac{i_1 + i_2}{2} - \frac{n + 2}{24}(i_2 - i_1)^2 + \cdots. \tag{11}$$

Substituting this value of i given by (11) in (10) gives

$$E(\max) = -\left[\frac{V' - V''}{n(i_2 - i_1)}\right]^{-1/(n+1)}\left(1 + \frac{1}{n}\right) + (1 + i_1) - \frac{V'(i_2 - i_1)}{V'' - V'} = \frac{n + 1}{8}(i_2 - i_1)^2 - \cdots. \tag{12}$$

The following theorems may be presented:

THEOREM VIII: *The maximum error in solving for an unknown rate in the* v^n *table occurs when* i *is slightly less than the mean of the table rates* i_1, i_2.
THEOREM IX: *The error due to linear interpolation in solving for an unknown rate in the* v^n *table is never more than* $\frac{1}{8}(n + 1)/(i_2 - i_1)^2$.

We may now state and prove a theorem in regard to the size of the error obtained in the v^n table and the $a_{\overline{n}|}$ table.

THEOREM X: *The error obtained in solving for an unknown rate by linear interpolation is always less in the* $a_{\overline{n}|}$ *table than in the* v^n *table.*

This theorem states that

$$i_1 + \frac{a - a'}{a'' - a'}(i_2 - i_1) - i < i_1 + \frac{V - V'}{V'' - V'}(i_2 - i_1) - i$$

where the errors are both positive, and

$$a' = \frac{1 - (1 + i_1)^{-n}}{i_1}, \qquad a = \frac{1 - (1 + i)^{-n}}{i}, \qquad a'' = \frac{1 - (1 + i_2)^{-n}}{i_2},$$
$$i_1 < i < i_2.$$

The inequality (8) may be reduced to

$$\frac{a - a'}{a'' - a'} < \frac{V - V'}{V'' - V'}$$

or,

$$a(V'' - V') + a'(V - V'') + a''(V' - V) < 0,$$

or

$$\frac{s}{A}\left(\frac{1}{A''} - \frac{1}{A'}\right) + \frac{s'}{A'}\left(\frac{1}{A} - \frac{1}{A''}\right) + \frac{s''}{A''}\left(\frac{1}{A'} - \frac{1}{A}\right) < 0. \tag{13}$$

Multiplying both members of (13) by $A'AA''$ gives

$$s(A' - A'') + s'(A'' - A) + s''(A - A') < 0$$

which is an expanded form of inequality (6) which was proved to be true.

By Theorem X *the maximum error given by (12) also applies to the $a_{\overline{n}|}$ table.*

Since the yield on a bond may be found approximately by interpolating in the $a_{\overline{n}|}$ table, the maximum error is given by (12).

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[1] It can be shown that the value of N obtained by interpolation is the exact value of n if simple interest is used for the fractional interest period involved.

[2] In W. L. HART, *Mathematics of Investment*, second ed. Boston, 1929, p. 244, a proof is given that the error is at most $\frac{1}{2}$ of the interest rate per period.

[3] The value of $n = n_1 + f$ ($f < 1$) obtained in the $a_{\overline{n}|}$ table has the following useful interpretation: f is the final payment due at the end of $n + 1$ interest periods.

[4] See THEODORE E. RAIFORD, *Mathematics of Finance*. Boston, 1945, p. 25, note, and W. L. Hart, *Mathematics of Investment*, third ed., Boston, 1946, p. 75 and p. 138. In these texts the following statement is made. Experience shows that it is safe to assume that a value of I found by interpolation is in error by not more than $\frac{1}{10}$ of the difference of the table rates used in the interpolation.

RECENT MATHEMATICAL TABLES

608[A, D, S].—G. H. GOLDSCHMIDT & G. J. PITT, "The correction of X-ray intensities for Lorentz-polarization and rotation factors," *Jn. Sci. Instrs.*, v. 25, Nov. 1948, p. 397–398. 20.2 × 27.2 cm.

There are two tables. **T. 1**, Inverse Lorentz-polarization factor as a function of $\rho = 2\sin\theta$; $(LP)^{-1} = \sin 2\theta/(1 + \cos^2 2\theta) = \frac{1}{2}\rho(4 - \rho^2)^{\frac{1}{2}}/(2 - \rho^2 + \frac{1}{4}\rho^4)$, for $\rho = [0(.01)2; 4D]$. **T. 2**, Rotation factor D as a function of ξ and ζ for equi-inclination conditions, $\xi = \zeta D/(1 - D^2)^{\frac{1}{2}}$; $D/(1 - D^2)^{\frac{1}{2}}$ is given, 2–3 decimal places, for $D = -1, .2(.1).9, .95, .975$.

Extracts from text

609[B].—LUDWIG ZIMMERMANN, *Vollständige Tafeln der Quadrate aller Zahlen bis 100 009 berechnet und herausgegeben.* Fourth edition, Berlin Grunewald, 1941, xix, 187 p. (*Sammlung Wichmann Fachbücherei für Vermessungswesen und Bodenwirtschaft*, v. 8.) 19.4 × 24.9 cm.

In the publisher's preface we are told that Zimmermann died 15 Aug. 1938. Compare *MTAC*, v. 2, p. 206–207; the errors of the third edition (1938), in T. III, there noted, here persist. The second edition was published at Liebenwerda in 1925; and the first in 1898.

Zimmermann was also the author of: (a) *Rechentafeln, grosse Ausgabe.* Liebenwerda, 1896, xvi, 205 p.; second ed., 1901; third ed., 1906; fourth ed., 1923, xxxix, 225 p. (b) *Rechentafeln, kleine Ausgabe.* Liebenwerda, 1895; fourth ed., 1926, xxv, 38 p. (c) *Tafeln für die*

Teilung der Dreiecke, Vierecke, und Polygone. Second enl. and impr. ed., Liebenwerda, 1896, 118 + 64 p. (d) *Die gemeinen oder briggschen Logarithmen der natürlichen Zahlen 1–10009 auf 4 Decimalstellen nebst einer Productentafel, einer Quadrattafel und einer Tafel zur Berechnung der Kathete und Hypotenuse und zur Bestimmung der Wurzeln aus quadratischen Gleichungen. Zum Gebrauch für Schule und Praxis.* Liebenwerda, 1896, 40 p. (e) *Mathematische Formelsammlung . . . zur Vorbereitung für das Einjährig-Freiwilligen-Examen. Essen,* Baedeker, 1910, iv, 55 p.

610[D, S].—J. D. H. Donnay & G. E. Hamburger, *Tables for Harmonic Synthesis, giving Terms of Fourier Series to one decimal at every millicycle tabulated for coefficients 1 to 100, and fiducial cosine values to eight Decimals.* Baltimore, The Johns Hopkins University, Crystallographic Laboratory, copyright 1948, [103] leaves. 21.6 × 28 cm. The leaves are enclosed in a strong ring binder. Purchasable from Professor Donnay at the Laboratory, $10.00.

The tables on 100 leaves give $10F \cos X$, for $F = 1(1)100$, $X = 1(1)1000$. The unit of angle, $2\pi/1000$, is called a millicycle (mC). Since $\sin X = \cos(X + 750)$, the table also gives $10F \sin X$. In all 200 000 values are thus available. They are represented by 25 000 entries arranged as follows: one page for every value of F and 250 entries on each page.

On leaf [103] is a table of $\cos X$, $X = [1(1)250; 8D]$; 1 mC = .36°. This is simply a table of extracts from Earle Buckingham, *Manual of Gear Design,* Section 1, 1935. [See *MTAC,* v. 1, p. 88–92]. The table on leaf [3] is equivalent to 1D from Buckingham's table and leaf [102] is equivalent to 3D from the same table.

A comparison of the 15D table of sines and cosines in F. Callet, *Tables Portatives de Logarithmes,* Paris, 1795, revealed two last-decimal unit errors in Buckingham, namely: cos 27 mC = .98564460 (not .98564459), and cos 232 mC = .11285638 (not .11285639).

The tables are intended to facilitate computations in problems of harmonic synthesis. In structural crystallography, for example, it may be used for the summation of Fourier Series representing either the electron intensity $\rho(xyz)$ at a point xyz, or the structure factor $F(hkl)$ of a reflection hkl.

Example: $$\rho(xyz) = \sum_h \sum_k \sum_l A(hkl) \cos 1000(hx + ky + lz)^{\mathrm{mC}} + \sum_h \sum_k \sum_l B(hkl) \sin 1000(hx + ky + lz)^{\mathrm{mC}}.$$

The required multiples of the trimetric coordinates x, y, z are tabulated once and for all. The angle $(hx + ky + lz)$ is calculated for every triplet hkl; in each case, the first three decimal places give the X of the table.

Alternately the three-dimensional triple sums may be replaced by one-dimensional single sums,

$$\sum_H A(H) \cos 1000HX^{\mathrm{mC}} + \sum_H B(H) \sin 1000HX^{\mathrm{mC}},$$

by letting $H = n^2h + nk + l$, where $n = 100$ (or 1000) according as xyz are given to two (or three) places, and

$$X = (n^2z + n^2y + nx)/n^3.$$

The basis for the second method is that evaluating the electron density along a (one-dimensional) row $[n^2nl]$ amounts, if n is sufficiently large, to evaluating it throughout the whole three-dimensional cell.

Extracts from text

Editorial Note: The millicycle angular unit here used is $\frac{2}{5}$ of a grade = .001 of a gone = .1 of a Cir. See *MTAC,* v. 1, p. 40–41.

611[E, H].—C. I. ROBBINS & R. C. T. SMITH, "A table of roots of $\sin z = -z$," *Phil. Mag.*, s. 7, v. 39, Dec. 1948, p. 1004–1005. 17.2 × 25.6 cm.

On writing $z = x + iy$ and separating into real and imaginary parts this equation is replaced by the pair of equations

$$f(x, y) = \cosh y + x/\sin x = 0, \qquad g(x, y) = \cos x + y/\sinh y = 0.$$

A 6D table of the first 10 non-zero roots in the first quadrant is given. These values were calculated by Newton's rule in the form

$$f(x + \delta x, y + \delta y) \approx f(x, y) + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} = 0,$$

$$g(x + \delta x, y + \delta y) \approx g(x, y) + \delta x \frac{\partial g}{\partial x} + \delta y \frac{\partial g}{\partial y} = 0.$$

Starting from values x, y these two equations determine δx, δy and thus improved approximations $x + \delta x$, $y + \delta y$ to the roots. This process was repeated until the corrections did not affect the eighth decimal. Consequently the roots are believed reliable to 6D. This table is similar to that of A. P. HILLMAN & H. E. SALZER, giving 6D values of the first 10 zeros of $\sin z - z$, *Phil. Mag.*, s. 7, v. 34, 1943, p. 575.

Extracts from text

EDITORIAL NOTE: In *MTAC*, v. 2, p. 60–61, Jan. 1946, MITTELMAN & HILLMAN published 7D values of the first four non-zero values of the zeros of $\sin z + z$. In 1940 J. FADLE published 5D values in *Ingenieur-Archiv*, v. 11, p. 129. See also *MTAC*, v. 1, p. 141, 50.

612[G, K].—S. M. KERAWALA & A. R. HANAFI, "Table of monomial symmetric functions of weight 12 in terms of power-sums," *Sankhyā*, v. 8, June, 1948, p. 345–359. 23 × 29.5 cm.

This table is an extension of previous tables of symmetric functions of weights less than 12 described in RMT **463** (*MTAC*, v. 3, p. 24). Each line of the table gives the coefficients of the polynomial in $S_1, S_2, \cdots, S_{12}$ which represents a given monomial symmetric function $\Sigma\, \alpha_1^{p_1}\alpha_2^{p_2} \cdots$ where $p_1 + p_2 + \cdots = 12$. Here S_k is the sum of the kth powers of the α's. For some reason the last line of the table has been omitted; that is to say, there is no expression given for the 12-th elementary symmetric function $\Sigma\, \alpha_1\alpha_2 \cdots \alpha_{12}$ as a polynomial in the S's.

D. H. L.

613[I].—HERBERT E. SALZER, *Table of Coefficients for Obtaining the First Derivative without Differences.* (NBS, *Applied Mathematics Series*, no. 2.) Washington, D. C., 1948, [ii], 20 p. 19.8 × 26 cm. For sale by the Superintendent of Documents, Washington, D. C., 15 cents. See *MTAC*, v. 3, p. 187–188.

The present set of tables is designed to expedite the numerical estimation of the values of the derivative of a function, $f(x)$, which has been approximated by means of the interpolation formula of LAGRANGE. In 1944 the NBSCL under the direction of Dr. A. N. LOWAN provided a large volume of the coefficients in the Lagrange approximation formula

$$f(x_0 + ph) \sim \sum_{i=-[\frac{1}{2}(n-1)]}^{[\frac{1}{2}n]} A_i^{(n)}(p) f(x_0 + ih)$$

where $[m]$ means the largest integer in m. (For a review see *MTAC*, v. 1, p. 314–315.)

If the derivative of both sides of this approximation is taken with respect to p there results:

$$f'(x_0 + ph) \sim \frac{1}{hC(n)} \sum_{i=-[\frac{1}{2}(n-1)]}^{[\frac{1}{2}n]} C_i^{(n)}(p) f(x_0 + ih).$$

The present work provides tables for the coefficients in this sum. To quote the author:

"$C_i^{(n)}(p)$, or simply C_i, is a polynomial in p of the $(n-2)$th degree, and $C(n)$, or simply C, is the least integer chosen, so that $C_i^{(n)}(p)$ will have integral coefficients $\cdots$. The accompanying table gives the exact values of these polynomials $C_i^{(n)}(p)$, for p ranging from $-[(n-1)/2]$ to $[n/2]$. For $n = 4$, 5, and 6, the polynomials $C_i^{(n)}(p)$ are tabulated at intervals of 0.01; for $n = 7$, they are tabulated at intervals of 0.1.

For $n = 3$, no table is needed, because we have the quite simple formula

$$f'(x_0 + ph) \sim \frac{1}{h}[(p - \tfrac{1}{2})f_{-1} - 2pf_0 + (p + \tfrac{1}{2})f_1]."$$

H. T. D.

614[I].—P. M. WOODWARD, "Tables of interpolation coefficients for use in the complex plane," *Phil. Mag.*, s. 7, v. 39, Aug. 1948, p. 594–604. 16.9 × 25.1 cm.

T. I: $C_2(p, q) = \frac{1}{6}pq(p^2 - q^2)$, for p and $q = [0(.05)1; 7D]$, with coupled second differences—exact to 8D; **T. II:** $C_4(p, q) = -[pq/(360)][4 + (p^2 - 3q^2)(3p^2 - q^2)]$, for p and $q = [0(.05)1; 7D]$, also with coupled second differences. Fourth differences, which never exceed 60, are not tabulated.

Extracts from text

615[L].—P. K. BOSE, "On recursion formulae, tables and Bessel function populations associated with the distribution of classical D²-statistic," *Sankhyā*, v. 8, Oct. 1947, p. 235–248. 22.7 × 29.3 cm.

On p. 247–248 are 6D tables of $e^{-x}I_0(x)$, $e^{-x}I_1(x)$ for $x = 16.08$, 16.2, 16.68, 16.92, 17., 17.04, 17.16, 17.4, 18, 18.48, 16.16(.16)18.88 [except 17.6, 18.4], 19, [200 other values], 49.2, 49.28, 49.44, 49.6, 49.68, 49.92, 50.

In *MTAC*, v. 1, p. 226, we have given references to tables of these functions in (a) BAASMTC, *Math. Tables*, v. 6, 1937, for $x = [16(.1)20; 8D]$, δ^2; and (b) BADELLINO, 1939, for $x = [20(1)50; 9D]$.

As to the earlier pages, J. W. TUKEY noted in *Math. Revs.*, v. 9, p. 620: "The distribution in question was found by R. C. Bose, *Sankhyā*, v. 2, 1936, p. 143–154. It can be put in the form

$$P(L) = \int_0^P x^{q+1}\lambda^{-q}e^{-\frac{1}{2}(x^2+\lambda^2)}I_q(x\lambda)dx, \quad q = \tfrac{1}{2}p - 1,$$

where $2L^2 = \bar{n}pD_1^2 = \bar{n}pD^2 + 2p$, $2\lambda^2 = \bar{n}p\Delta^2$ and Δ^2 and D^2 are the population and estimated squared distances of two p-variate samples whose harmonic mean size is $\bar{n}$." Values of L are tabulated for $P = .99$, .95, .05 and .01, $p = 1(1)10$, that is, $q = -\frac{1}{2}(\frac{1}{2})4$, and $\lambda = 0(.5)3(1)6$, 8, 12(6)24, 36, 54, 72, 108, 216, 432 to $2D$.

R. C. A.

616[L].—C. H. COLLIE, J. B. HASTED, & D. M. RITSON, "The cavity resonator method of measuring the dielectric constants of polar liquids in the centimetre band," Phy. Soc., London, *Proc.*, v. 60, 1948, p. 71–82. 17.8 × 25.9 cm.

On p. 78–79 is a 4D or 4S table of the real and imaginary parts of $z^{-1}J_1(z)/J_0(z)$, $z = x - iy$, for $x = .5(.1)2$, $y = 0(.1)1$, $y \leqslant x$. On p. 77 is a graph of the imaginary part of the function.

617[L].—A. GHIZZETTI, "Tavola della funzione euleriana $\Gamma(z)$ per valori complessi dell' argomento," Accad. Naz. Lincei, *Atti*, *Rend.*, s. 8, v. 3(2), 1947, p. 254–257. 18 × 26.5 cm.

Table of $\Gamma(x + iy)$, $x = 4(.1)5$, $y = 0(.1)1$, to 5S. For interpolation the marginal values $3.9 + .2ni$, $5.1 + .2ni$, $4 - .1i + .2n$, $4 + 1.1i + .2n$, $n = 0(1)5$. See further *Math. Revs.*, v. 8, p. 619 (S. C. VAN VEEN).

618[L].—S. GOLDMAN, *Frequency Analysis, Modulation and Noise.* New York, McGraw-Hill, 1948. "Appendix F. Table of Bessel Functions of the first kind of constant integral argument and variable integral order," p. 421–427.

This is a Table of $J_n(x)$ for $x = [1(1)33; 4\text{–}5S]$, $n = 0(1)N - 1$, $15 \leq N \leq 37$. It is said to be of particular use in determining the sideband magnitudes in frequency and phase modulation. That part of the Table $n = 1(1)29$, $15 \leq N \leq 35$ is evidently abridged from JAHNKE & EMDE's expanded abridgment (1938) of MEISSEL's table of 1895, because J. & E.'s error in $J_9(21)$ is faithfully copied. The remaining part of the table, p. 427, was abridged from the HARVARD COMPUTATION LABORATORY's Bessel Function calculations. See N99, no. 7.

R. C. A.

619[L].—R. E. GREENWOOD & J. J. MILLER, "Zeros of the Hermite polynomials and weights for Gauss' mechanical quadrature formula," Amer. Math. Soc., *Bull.*, v. 54, Aug. 1948, p. 765–769. 15.2 × 24.1 cm.

The Hermite polynomials are defined by the relation

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} d^n(e^{-x^2})/dx^n \\ &= (2x)^n - n(n-1)(2x)^{n-2}/1! + n(n-1)(n-2)(n-3)(2x)^{n-4}/2! - \cdots. \end{aligned}$$

Some writers, including many statisticians, prefer to use

$$h_n(x) = e^{\frac{1}{2}x^2} d^n(e^{-\frac{1}{2}x^2})/dx^n$$

as the defining relation for Hermite polynomials. The relation between these two sets of polynomials is given by

$$H_n(x) = (-2^{\frac{1}{2}})^n h_n(2^{\frac{1}{2}}x).$$

The approximate numerical integration formula for functions $f(x)$ on the infinite range $(-\infty, +\infty)$ with the weight function $\exp(-x^2)$ is

$$(1) \qquad \int_{-\infty}^{\infty} e^{-x^2} f(x) dx \backsimeq \sum_{i=1}^{n} \lambda_{i,n} f(x_{i,n})$$

where the set $\{x_{i,n}\}$ is the set of roots defined by $H_n(x) = 0$, and where the set $\{\lambda_{i,n}\}$ is given by[1] $\lambda_{i,n} = \pi^{\frac{1}{2}} 2^{n+1} n!/[H_n'(x_{i,n})]^2$. If $f(x)$ is a polynomial of degree $(2n - 1)$ or less, integration formula (1) is exact.[2]

The zeros $\{X_{i,n}\}$ for the polynomials $h_n(x)$, for $n = [1(1)27; 6D]$ have been tabulated by SMITH.[3] The corresponding zeros are given by $x_{i,n} = 2^{-\frac{1}{2}} X_{i,n}$.

Tables, 9–12D or S, are given of $\{x_{i,n}\}$ and of Christoffel numbers $\{\lambda_{i,n}\}$ for $n = 1(1)10$, $i = 1(1)5$.

Extracts from text

[1] G. SZEGÖ, *Orthogonal Polynomials*, Amer. Math. Soc., *Colloquium Publs.*, v. 23, 1939, p. 344.

[2] SZEGÖ, *loc. cit.*, chap. XV; and C. WINSTON, "On mechanical quadratures formulae involving the classical orthogonal polynomials," *Annals Math.*, s. 2, v. 35, 1934, p. 658–677.

[3] E. R. SMITH, "Zeros of the Hermitian polynomials," *Amer. Math. Mo.*, v. 43, 1936, p. 354–358. [EDITORIAL NOTE: see *MTAC*, v. 1, p. 152–153; for varying definitions of $H_n(x)$, see here, and v. 1, p. 50; v. 2, p. 25, 30; v. 3, p. 26, 167. Greenwood & Miller give no reference to the table of REIZ, reviewed in *MTAC*, v. 3, p. 26.]

620[L].—U. S. Navy, Naval Research Laboratory, *Extended Tables of Fresnel Integrals.* Boston, Mass., 470 Atlantic Ave., 1948, 7 + [6] hektographed leaves, 26.6 × 20.3 cm.

The Tables involve $C(u)$, $S(u)$, $c_1(u) = \int_\infty^u \cos(\frac{1}{2}\pi t^2)dt = C(u) - \frac{1}{2}$, $s_1(u) = \int_\infty^u \sin(\frac{1}{2}\pi t^2)dt = S(u) - \frac{1}{2}$.

Table I is of $C(u)$, $S(u)$, $[C(u)]^2$, $[S(u)]^2$, $\{[C(u)]^2 + [S(u)]^2\}^{\frac{1}{2}}$, $[c_1(u)]^2$, $[s_1(u)]^2$, $\{[c_1(u)]^2 + [s_1(u)]^2\}^{\frac{1}{2}}$, for $u = [0(.1)20; 4D$ or $4S]$.

Table II is of $C(u)$ and $S(u)$, for $u = [8(.02)15.98; 4D]$.

Among tables of this kind are those of $C(u)$ and $S(u)$ for $u = [0(.1)8.5; 4D]$ in JAHNKE and EMDE, 1945, American edition, which are in agreement with the tables under review except for the Navy error in $S(7.3)$, where *for* .3189, *read* .5189.

R. C. A.

621[L, M].—J. C. JAEGER, "Repeated integrals of Bessel functions and the theory of transients in filter circuits," *Jn. Math. Phys.*, v. 27, Oct. 1948, p. 210–219. 17.3 × 25.3 cm.

$Ji_{n,r}(t) = \int_0^t dt \int_0^t dt \cdots \int_0^t J_n(t)dt$, **T. I** is of $2^r Ji_{n,r}(t)$, for $r = 1(1)7$, $t = [0(1)24; 8D]$. $\Phi_n(t) = \int_0^\infty J_0[2v(ut)]J_n(u)du$, $J_n(t) = \int_0^\infty J_0[2v(ut)]\Phi_n(u)du$, **T. II–III** give 4D values of $\Phi_n(t)$, and $\Phi_n'(t)$, for $n = 1(1)7$, $t = 0(1)24$. $\psi_n(t) = \frac{1}{2}[\Phi_{2n}(t) + \Phi_{2n+2}(t)]$, **T. IV** gives 5D values of $\psi_n(t)$, for $n = 0(1)2$, $t = 0(1)24$.

Extracts from text

622[L, Z].—P. I. ZUBKOV, "Primenenie universal'nogo raschetnogo stola peremennogo toka dli͡a tabulirovanii͡a otnoshenii͡a modifit͡sirovannykh funkt͡siĭ Besseli͡a" [The application of a universal alternating current computer to the tabulation of ratios of modified Bessel functions], Akad. N., SSSR, *Izvestii͡a, Otdelenie tekhnich. n.*, 1948, p. 489–498.

The computer mentioned in the title is not described in the present paper. It appears to be a sort of AC network analyzer with adjustable resistances, inductances and capacities. The author mentions negative resistances from which one would infer the existence of two or more amplifiers probably with feedback facilities.

The functions referred to are the familiar $zI_{n-1}(z)/I_n(z)$ whose well known continued fraction development is exploited by the computer. Just how many terms of this development the machine can use is not revealed. The values of n considered are $n = \pm .1$. The complex number z is of the form $i^{\frac{1}{2}}x$ where x is real. Tables are given to 4S of the real and imaginary parts of the functions for $x = 0(.2)2$. These tables were computed by hand. Corresponding readings taken from the machine bear no superficial resemblance whatever. With a certain amount of careful guesswork, values of the functions can be derived from the readings. The agreement is said to be within 2 percent.

D. H. L.

623[M].—E. C. BULLARD & R. I. B. COOPER, "The determination of the masses necessary to produce a given gravitational field," R. Soc. London, *Proc.*, v. 194A, 1948, p. 332–347.

$\lambda(x) = 2(\sigma/\pi)^{\frac{1}{2}} \cos 2\sigma x \exp[\sigma(1 - x^2)] - (1/\pi)$, $\psi(x, \sigma^{-\frac{1}{2}})\psi(x, \eta) = \eta^{-1}\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}(1 + y^2)^{-1} \times \exp[-\overline{(x - y/\eta)^2}]dy$; $\Lambda(r) = r\int_0^\infty e^{p}e^{-p^2/4\tau}J_0(pr)pdp$.

T. 1 gives values of $\lambda(x)$ for $\sigma = 1(4D)$, $4(3D)$ and $x = 0(.1)5$, with modified second differences; also for $\sigma = 1(4D)$ the values of $\int_0^x \lambda(x_1)dx_1$; also 4D values of $\Lambda(r)$ and $\int_0^r \Lambda(r_1)dr_1$, for $\tau = 1$. Graphs of $\lambda(x)$ for $\sigma = 1$ and $\Lambda(r)$ for $\tau = 1$. The column giving $\lambda(x)$ for $\sigma = 4$ cannot be interpolated.

Beyond the limits of **T. 1**, the following expressions will give results accurate to four places of decimals:

$$\pi\lambda(x) \sim -1/x^2 - (3/2\sigma - 1)/x^4, \quad \pi\int_0^x \lambda(x_1)dx_1 \sim \tfrac{1}{2}\pi + 1/x + (3/2\sigma - 1)/3x^3;$$
$$\Lambda(r) \sim -1/r^2 - 3(3/2\tau - 1)/2r^4, \quad \int_0^r \Lambda(r_1)dr_1 \sim 1 + 1/r + (3/2\tau - 1)/2r^2.$$

T. 2, 3D values of $\int_{x_1}^{x_2}\lambda(x)dx$, for intervals 0 to .25; .25 to .5; .5 to 1; 1 to 1.5; 1.5 to 2; 2 to 3; 3 to 4; 4 to 5; 5 to 10; 10 to 20; 20 to ∞.

T. 3, 1D values of $\int_{r_1}^{r_2}\Lambda(r)dr$, for intervals 0 to .23; .23 to .34; .34 to .44; .44 to .54; .54 to .65; .65 to .83; .83 to 1.35; 1.35 to 1.59; 1.59 to 2.12; 2.12 to 2.54; 2.54 to 3.4; 3.4 to 5.05; 5.05 to 10.02; 10.02 to ∞.

Extracts from text

624[M].—H. G. Hay, & Miss N. Gamble, "Five-figure table of the function $\int_0^\infty e^{-zy}\cdot Ai^2(y - j_1)dy$ in the complex plane," *Phil. Mag.*, s. 7, v. 39, Dec. 1948, p. 928–946. 17.2 × 25.6 cm.

Except for two references to recent literature this paper is simply an edition for the general public of the report in Nov. 1946, which we have already reviewed in *MTAC*, v. 2, p. 344–345. In the reprint a small misprint has been introduced, p. 931, l. −1; *for* 3.3, *read* 3.2. We are told that a full description of the method used in the computation is due to G. G. Macfarlane, "The application of a variational method to the calculation of radio wave propagation curves for an arbitrary refractive index profile in the atmosphere," Phys. Soc. London, *Proc.*, v. 61, July 1948, p. 48–59. A reference is also given to P. M. Woodward, *Phil. Mag.*, s. 7, v. 39, Aug. 1948, p. 594–604 (RMT **614**), for his method of using coupled differences in bivariate interpolation for a function of a complex variable. A single-page sample of Woodward's tables appeared in the 1946 report of Hay & Gamble.

R. C. A.

625[Q].—Paul Herget, *The Computation of Orbits.* Published privately by the author, University of Cincinnati, and lithographed by Edwards Brothers, Inc., Ann Arbor, Michigan, 1948, ix, 177 p. including 24 p. of tables. Light cardboard binding, 22 × 28 cm. $6.25.

There are so few works on the computation of orbits and perturbations in English, while foreign works are so thoroughly out of print, that Herget's highly condensed but surprisingly comprehensive treatment of the subject must perforce be welcomed wherever advanced astronomy is studied. Some devotees of Leuschner's modification of Laplace's method will be stunned by its omission, but Herget has treated the various methods of computing preliminary orbits with his own independent and experienced approach. His inclusion of numerical differentiation and integration, special perturbations, and the application of Hansen's method of general perturbations as well as the computation of preliminary and corrected orbits in so small a volume is remarkable. Compactness is gained by the use of vector notation, unfortunately at times because of the difficulty in distinguishing vector from scalar font. The few, highly specialized, tables are also extremely compact and most of them require second difference interpolation. Most of these tables are available elsewhere in various forms.

The book should contribute appreciably to the preservation of orbit work in present-day astronomy.

Table **I** (3D) gives constants for correcting rectangular coordinates of the sun from an origin at the center of the earth to the position of the observer. Arguments: astronomical latitude and selected Observatories.

T. **II** is a critical table (4D) of Everett second differences interpolating coefficients. This interpolating formula is often useful in tables where the fourth differences can be neglected, and especially where also the second differences are tabulated.

If S is the chord between positions in a parabolic orbit at times t_i and t_j and solar dis-

tances r_i and r_j, then T. III (7D) gives $S/(r_i + r_j) = \eta\zeta$ as a function of $\eta = 2k(t_j - t_i)/(r_i + r_j)^{\frac{3}{2}}$, 0(.01).6; it gives also $\bar{y}$, the ratio of the sector area to triangle area, for the same argument. In adjacent columns T. III (7D) gives the solution y of the equation

$$y^3 - y^2 - hy - h/9 = 0$$

with argument h = 0(.01).6, and also $\Delta\bar{y}_0$ used in numerical approximating. These tabulated quantities are useful in the Gaussian method for calculating parabolic orbits.

T. IV (7D) gives, both for ellipse and hyperbola, two highly specialized quantities used in solving Lambert's equation and used in GAUSS's method mentioned above. Included also in T. IV is the function f, (6D), where

$$fq = 1 - (1 + 2q)^{-\frac{3}{2}}$$

with argument q = − .03(.001) + .03. This function occurs in ENCKE's method of special perturbations.

In relating true anomaly with time in a nearly parabolic orbit, either elliptic or hyperbolic, certain auxiliary quantities, A, B (8D), C (7D), and D (7D), given with argument A = 0(.0001).3, in T. V reduce the numerical work involved.

T. VI gives the interpolating coefficients to be used in tables of double and single numerical integration in an "Everett" form, where only even-order differences "on the line" need be tabulated. The argument is n = 0(.001)1, the fraction of the interval, for the coefficients (6D and 7D) of the second and lower order differences. Coefficients (5D) of the fourth and sixth differences are given at the end of the main table.

Herget states: T. VII "is an 'optimum-interval' table which gives $1/r^3$ with the argument r^2. The interpolating formula is

$$F(r^2) = F_0 - N(D_1 - ND_2)$$

where N consists of all of the portion of r^2 which is not printed in full-size type in the r^2 column." The quantities F_0, D_1, and D_2, to seven significant figures (8D or 9D), are given with argument r^2 = 4(.01)40, where the second or the first and second decimals are small-size type. The table covers only three pages!

[T. I is an abbreviation of a table by E. C. BOWER in Lick Observatory, *Bull.*, v. 16, 1932, p. 41; parts were given earlier in the British *Nautical Almanac.*—T. II, VI, see J. D. EVERETT, BAAS, *Report 1900*, p. 648–650; E. T. WHITTAKER & G. ROBINSON, (a) *A Short Course in Interpolation.* London, 1923, p. 40; (b) *The Calculus of Observations.* . . . 1924, p. 40.—T. III. For $\Delta\bar{y}$, see RAS, *Mon. Not.*, v. 90, 1930, p. 814; $\eta\zeta$ is condensed from T. 26 in J. BAUSCHINGER, *Tafeln zur theoretischen Astronomie*, second ed. by G. STRACKE. Leipzig, 1934.—T. V, Logarithmic equivalents are given by A. MARTH, *Astron. Nachrichten*, v. 43, 1856, cols. 121–134.—T. VII, see, for example, British *Nautical Almanac* 1933, Table X, "Planetary coordinates for the years 1800–1940."]

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EDITORIAL NOTE: On Jan. 6, 1949, we received from Professor Herget a "First List of Errata and Addenda" containing 34 entries.

626[U].—ISTITUTO IDROGRAFICO DELLA MARINA, *Tavole H per il calcolo delle Rette d'Altezza.* (Publication No. 3118.) Genoa, 1947. xxv, 30 p. and 2 charts. 17.1 × 24 cm. 600 lire.

This attractive small volume, bound in tan cloth, was prepared, according to its preface, to replace "Tavole F" which was out of print. Although no information on the point is included in this volume, one may reasonably suspect that "Tavole F" was an Italian reprint of the German "F-Tafeln" [*MTAC*, v. 2, p. 81–82]. In any case, the preface states specifically that this volume, save for a few additions and changes in the explanatory material and a few examples, is a direct copy of *Tavole A B per le rette di altezza*, a publication of the

Istituto Geografico Militare, which was in turn made up of the table of S. OGURA, and the azimuth diagram of A. RUST.

Thus this table may be compared to H.O. no. 208 (DREISONSTOK) [*MTAC*, v. 1, p. 79–80], with the inclusion of Rust's Azimuth Diagram permitting a reduction in the amount of material in Tables I and II. The astronomical triangle is divided into two right triangles by a perpendicular dropped from the zenith upon the hour circle of the celestial body; N is the length of this perpendicular and K is the declination of its foot. In the usual American notation, in which L, d, t, h, Z are the latitude of the observer, the declination, hour-angle, altitude and azimuth of the celestial body, the formulae used to compute the values in Table I are:

$$\tan K = \tan L \sec t, \qquad \text{and} \qquad \cos N = \sin L \csc K.$$

Table I is a double-entry table with vertical argument, $L = 0(1°)65°$, and horizontal argument, 12 values to a page, $t = 0(1°)180°$; for each argument pair are tabulated the values of $A = 10^5 \log \sec N$ to 0.1 for $t = 0(1°)20°$ and $160°(1°)180°$ and to the nearest integer for intermediate values of t, and of K to $0'.1$ for $t = 0(1°)180°$.

Table II is one of the values to the nearest 0.1 of $10^5 \log \sec (K - d)$ for $(K - d)$ $0(1')10°$ and to the nearest integer for $(K - d)$ $10°(1')80°$, and of $10^5 \log \csc h$ to the nearest integer for $h = 10°(1')80°$ and to the nearest 0.1 for $h = 80°(1')90°$.

h and Z may be obtained from these tables by the use of the formulae

$$10^5 \log \csc h = 10^5 \log \sec N + 10^5 \log \sec (K - d),$$
$$10^5 \log \csc Z = 10^5 \log \csc t + 10^5 \log \sec d - 10^5 \log \sec h,$$

or Z may be obtained from the Rust diagrams.

These tables have the common disadvantage of having the values corresponding to a single latitude scattered over all of the pages of Table I. Also they are useful only between 65°S and 65°N latitudes. They have the great advantages of simplicity and compactness and they can be used for all four of the basic problems of celestial navigation: the computation of altitude, of azimuth, the identification of stars, and the computation of great circle courses.

It might be added that the end papers in the front and back are very nice examples of a repetitive pattern involving a number of objects common in the navigator's world; these papers have no particular value in navigation but they do add to the attractiveness of the small volume. There are doubtless many navigators who would rather have some of the frequently used small tables, refractions, etc., on the inside covers and reserve the fine printing for the end papers of volumes on art and history.

To obtain an estimate of the accuracy of the tabular values, 1040 values of A in Table I were examined and 171 values were found to be in error. However, all of these errors were rounding-off errors of one unit in the last place given, except for a single error of two units. This would indicate that *Tavole H* is slightly more accurate than H.O. no. 208 (DREISONSTOK).

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627[U].—LAPUSHKIN, *Morekhodnye Tablitsy 1943g* [Nautical Tables for the year 1943], Leningrad and Moscow, Hydrographic Administration, Military and Maritime Fleet, USSR, 1944. lxiii, 245 p. 16.5 × 25.7 cm. + 13 cards 15.2 × 24.9 cm.

This collection of tables was designed for the use of the surface navigator and is based upon an earlier publication of the same name, edited by V. V. AKHMATOV and published in 1933, with second and third editions appearing in 1934 and 1939. It is somewhat similar to, but rather more extensive than, the tables in BOWDITCH, *American Practical Navigator*.

There are 57 pages of explanation followed by 59 tables, not counting those on the cards,

found in a pocket on the inside of the back cover. Actually tables 44 to 59 inclusive are almost exclusively conversion tables, time to arc and conversely, inches to millimeters to millibars, degrees Fahrenheit to degrees Centigrade, etc.

Tables 1–6 are 4D: 1–2, logarithms and antilogarithms; **3–4**, addition and subtraction logarithms; **5–6**, natural values for $0(10')90°$, and logarithmic values for $0(1')90°$, of the six trigonometric functions.

T. 7 is a collection of 3D tables corresponding to **T. 1–6.**

T. 8 gives the 4D values of $\log \sin^2 \frac{1}{2}t$ and $\sin^2 \frac{1}{2}t$ for $t = 0(1')180°$. $\sin^2 \frac{1}{2}t$ will be recognized as the familiar haversine. **T. 9** and **11** give the corrections to $0'.1$ to observed altitudes 5° to 90° of the lower limb of the sun and of stars to take care of refraction, height of eye 0, 10(2)60 feet, and semi-diameter of the sun, the latter in **T. 9** only. **T. 9a** gives the additional correction necessary to take into account the variable semi-diameter of the sun and **T. 10** tabulates the values of the diameter of the sun on selected dates of the year.

T. 12 and **13** give the corrections to $0'.1$ to be applied to the altitudes of the lower and upper limbs of the moon for refraction, semi-diameter, parallax and height of eye 20 feet. **T. 12a–13a** are identical and provide the corrections necessary to take into account heights of eye other than 20 feet, 8(2)60 feet. **T. 14** gives the dip of the horizon to $0'.1$ for height of eye 0(0.5)3(1)18 meters; it is probable that the user will find the change of units, from feet in **T. 12a** and **13a** to meters in **T. 14**, confusing. One wonders also whether 20 feet will not be rather a small height of eye for the average sea-going vessel. **T. 15** provides the corrections to be applied to an altitude measured from a shore line rather than from a visible horizon.

T. 16 yields the corrections to $0'.1$ for refraction for low altitudes $-10'(2')1°30'(10')5°$, as well as for other altitudes $5°(20')10°(30')12°(1°)25°(2°)37°$, $40°(5°)80°$, 90° for a standard temperature of 10°C. and a standard barometric pressure of 760 mm. of mercury. **T. 16a–16b** give the corrections to be applied to the values taken from **T. 16** for temperatures $-20°(5°)40°$ and barometric pressures 720(5)780 mm of mercury respectively.

T. 17–19 are intended for use in determining the correction to the meridian of altitudes observed near the meridian. The formula used is

$$C = (200 \sin^2 \tfrac{1}{2}t)/K \text{ arc } 1' - \tfrac{1}{2}C^2 \tan H \text{ arc } 1'$$

where $K = 100 \tan L - 100 \tan d$, and H is the approximate meridian altitude. **T. 17** is a critical table of values of $100 \tan x$ for x, 0 to 90°, allowing one to evaluate K quite easily. **T. 18** is a double-entry table providing the first (and often the only significant) term to $0'.1$ with arguments K and t. **T. 19** yields the second term to $0'.1$ with arguments, the approximate value of the first term and the approximate meridian altitude.

T. 20 gives the range in hours and minutes for the use of near-meridian altitudes with arguments latitude $0(5°)40°(4°)60°(2°)80°$ and declination, same name and opposite name, $0(5°)20°$, 24°. **T. 21**, intended to be used in correcting the time of culmination, gives values of $15.28 \tan x$ to .01 for $x = 0(1°)79°$. **T. 22** gives the correction to .01 to be applied to an altitude measured near the meridian to obtain the corresponding meridian altitude.

T. 23–24 give the hour angle to the nearest minute of time and the altitude to $0°.1$ of a celestial body on the prime vertical with declination $1°(1°)24°(2°)52°(4°)60°$ at a point in latitude $1°(1°)40°(2°)80°$. **T. 25–26** give the change in altitude to $0'.1$ in one minute of time and the interval of time to $0^s.1$ corresponding to a change in altitude of 1′, for latitudes $0(10°)80°$ and azimuths $5°(5°)50°(10°)90°$.

T. 27, well hidden two thirds of the way through the volume shares with **T. 8**, one third of the way through, honors in importance as a navigating table. It occupies 23 pages and is intended to be used with the formulae given below in the usual American notation where t, d, L, Z and h are the hour angle and declination of the celestial body, the latitude of the observer and the azimuth and altitude of the celestial body:

$$T(K) = T(d) + S(t),\ T(Z) = T(t) - S(K) + C|K - L|,\ T(90° - h) = T|K - L| + S(Z).$$

The tabulated quantities, each given to the nearest integer, are:

$$C(x) = 2(10)^4 \log \csc x, \quad S(x) = 2(10)^4 \log \sec x, \quad T(x) = 2(10)^4 \log \tan x + 70725,$$

for $x = 0(1')90°$. These formulae correspond to a division of the astronomical triangle into two right spherical triangles by a perpendicular dropped from the celestial body upon the meridian. K is the declination of the foot of the perpendicular.

T. 28–29 give the apparent azimuth angle to 0°.1 of the rising and setting upper limb of the sun for latitudes 0(5°)20°(1°)75° and declination, same name as latitude and opposite name, 0(1°)24°.

T. 30–33 give respectively the change in longitude resulting from a 1′ change in latitude, the change in latitude corresponding to a 1^s change in time, and the changes in latitude and in time for a 1′ change in altitude. **T. 34** provides the values to 0′.1 of the difference of latitude and departure corresponding to a distance 0(1)100 nautical miles for course angles 1°(1°)90° [courses 1°(1°)360°]. **T. 35** gives the difference in longitude to .01 corresponding to a departure 1(1)9, 100 and a mid-latitude 0(1°)30°(0°.5)60°(0°.2)70°(0°.1)81°. **T. 36** is one of meridional parts based on BESSEL's formula; values are given to .1 for latitudes 0(1′)89°59′. **T. 37, 37a–37b** are for navigation along an arc of a great circle; they provide the latitude L of a point on the great circle path corresponding to a longitude λ as a solution of the equation:

$$\tan L = \sin (\lambda - \lambda_0) \tan C_0,$$

where λ_0 is the longitude of the nearer point of the path on the equator and C_0 is the course along the great circle at that point. **Table 38** gives the distances to .01 of an object from two points by two bearings measured with respect to the ship's course at these two points, and the distance to .01 from the point where the second bearing was taken to the point of closest approach to the object, each distance being given in terms of the distance run between the first and second bearings.

T. 39a is a double-entry table giving the speed of sound in sea water to the nearest integer in meters per second corresponding to a salinity of 0(5)40 per cent and a temperature of 0(5)30 degrees Centigrade. **T. 39b** is another double-entry table giving the correction to be applied to the depth 5, 10(10)500 meters found by an echo-sounding device when the speed of sound varies from the standard for the device by 5(5)100 meters per second.

Turning to the cards which are contained in a pocket on the inside of the back cover, Nomogram (N.) **I** gives the correction of an altitude to the meridian; it may be used instead of table 18. **N. II** is designed to permit the plotting of a position line from an azimuth observation; the example given to illustrate the use of the nomogram appears to be in error. If one uses $h = 19°$ instead of 10° as given, one obtains the answer given. The numerals on this particular nomogram are almost illegible, even in a good light. **N. III** provides azimuth angle from altitude and conversely, when local hour angle and declination are known. **N. IVa** yields the distance at which an object of known height above sea level can be seen by a person whose height of eye above the ocean is known. **T. IVa** and **IVb** give the distance to the visible horizon for heights of eye in feet, 0 to 23000, and in meters, 0 to 5100.

T. V gives the distance in miles to .01 with arguments, minutes of time elapsed 1(1)10 and speed in knots 1(1)60. **T. Va** gives minutes of time to .01 with arguments miles 1(1)10 and speed in knots 1(1)60. This latter table appears to be superfluous and impractical. All of the information in it likely to be of value is contained in **T. V**. One finds difficulty in imagining a circumstance where one will need to know the time to a hundredth of a minute required to travel an integral number of miles at an integral number of knots. However the criticism of superfluity can be levelled at a number of other tables in this volume.

T. VI–VIIb and **N. VIIa** are for computations made to allow for the ship's log and currents. **T. VIII** and **N. VIIIa** provide corrections to the ship's course. There are five tables, **IXa, IXb**, etc., yielding the distances of objects from observed vertical angles. **T. X** is provided for computation involved in manoeuvering.

The volume is well bound in fabrikoid and the paper is rather better than was formerly found in Russian publications.

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MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **609** (Zimmerman), **610** (Buckingham), **612** (Kerawala & Hanafi), **618** (Goldman), **620** (U. S. Navy), **624** (Hay & Gamble), **625** (Herget); N **99** (Bertrand, Davis & Kirkham, Gray & Mathews, Meissel).

149.—E. P. ADAMS, *Smithsonian Mathematical Formulae* . . ., First reprint, Washington, D. C., 1939. See also *MTAC*, v. 1, p. 191; v. 2, p. 46, 353; v. 3, p. 314.

P. 260, at $r = 45$, *for* 78689 . . ., *read* 0.78689 . . . .
p. 260, last line, *for* 90°r, *read* 90° $- r$.

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150.—S. P. GLAZENAP, *Matematicheskie i Astronomicheskie Tablit͡sy*. Leningrad 1932, p. 214–215.

Glazenap states that in K(86°48′) *for* 4.2744, *read* 4.2746. This is erroneous; 4.2744 is correct. The result is given correctly to 5D by H. B. DWIGHT in *Electrical Engineering*, v. 54, 1935, p. 711: 4.27444. By two methods I deduced the approximation K(86°48′) = 4.27444 35354 98331 19349 41.

This error of Glazenap has been twice reprinted in *MTAC*, namely: v. 1, p. 198, and v. 3, p. 268, and the reference on the latter page to corresponding errors in the first three editions of JAHNKE & EMDE is therefore incorrect; and further there is now an error at this point in the 1945 edition of Jahnke & Emde, as noted in *MTAC*, v. 3, p. 267.

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151.—E. JAHNKE & F. EMDE, *Tables of Functions*. New York 1945. Supplement. See *MTAC*, v. 3, p. 41.

P. 5, l. 68^{13}, *for* 8392, *read* 8492;
l. 68^{14}, *for* 14067, *read* 14267;
p. 56, l. 7, *for* d Ar Ctg, *read* d Ar Ctg x.

FRANK HARRISON

EDITORIAL NOTE: In the 1933 and 1938 editions, the corresponding corrections of p. 5 have been already noted, *MTAC*, v. 1, p. 397.

152.—T. L. KELLEY, *The Kelley Statistical Tables*, 1948; see *MTAC*, v. 3, p. 301.

I call attention to three errors in my *Tables*, namely:

p. 6, l. 21, *for* $-2t_{-2}$, *read* $+2t_{-2}$;
p. 7, l. 14, *for* $+u$, *read* $-u$;
p. 123, l. 3, $p = .9251$, the corresponding x, *for* 02 3827, *read* 1.4402 3827.

TRUMAN L. KELLEY

153.—H. W. RICHMOND, "Notes on a problem of the 'Waring' type," London Math. Soc., *Jn.*, v. 19, 1944, p. 38–41.

On p. 41, 1919 is erroneously listed among integers that are not the sum of four tetra-

hedral numbers, $n(n+1)(n+2)/6$, $n > 0$. The error is evident from the relation 1919 $= 816 + 816 + 286 + 1$. The only integers between 1000 and 2000 which are not the sum of four tetrahedral numbers are 1007, 1117, 1118, 1153, 1227, 1233, 1243, 1314, 1382, 1402, 1468, 1478, 1513, 1523, 1578, 1612, 1622, 1658, 1678, 1693, 1731, 1738, 1742, 1758, 1767, 1803, 1858, 1907, 1923, and 1933. Those less than 1000 have been given in the writer's "On numbers expressible as the sum of four tetrahedral numbers," London Math. Soc., *Jn.*, v. 20, 1945, p. 3.

H. E. SALZER

154.—BAASMTC, *Mathematical Tables*, v. 1, second ed., 1946; first ed., 1931. See *MTAC*, v. 2, p. 122–123.

The 12D or 10D tables of polygamma functions appearing here, p. 42–59, were compared with corresponding tables in H. T. DAVIS, *Tables of the Higher Mathematical Functions*, v. 1, 1933, p. 291–349, and v. 2, 1935, p. 27–130. This comparison failed to reveal any discrepancies in the tables of the trigamma and tetragamma functions, but it did show four discordant entries in the tables of the digamma (or psi) function and eight discrepancies in the tabulated values of the pentagamma function.

These questionable data were recalculated to at least 22D, and the resulting approximations are as follows:

x	$d \ln (x!)/dx$
13.2	2.61761 76236 89490 85323 44
15.0	2.74101 33283 27460 36838 67
15.3	2.76017 67302 88333 27815 12
16.0	2.80351 33283 27460 36838 67
	$d^4 \ln (x!)/dx^4$
0.40	1.82025 90339 47094 48138 65
24.4	0.00012 94440 44696 40467 30
26.4	0.00010 26770 36526 88073 24
29.4	0.00007 47779 76500 04736 67
39.4	0.00003 14756 68589 92571 28
42.4	0.00002 53244 58663 16147 29
48.4	0.00001 71006 50514 13644 04
50.4	0.00001 51632 69520 59975 27

From these more accurate data it may be concluded that the BAASMTC table of the digamma function contains a rounding error at $x = 13.2$, whereas at least three last-figure errors exist in Table 9, v. 1 of Davis's work. The three erroneous values correspond to $x = 16.00, 16.30$, and 17.00 in the notation of Davis, who tabulates $\Psi(x)$, which is equivalent to $d \ln (x-1)!/dx$ in the notation adopted by BAASMTC and retained in this note. It should be noted that in Table 10, v. 1, p. 348 Davis gives correct 16D values of $\Psi(16.0)$ and $\Psi(17.0)$. The discrepancies in the tables of the pentagamma function are all attributable to last-figure errors, each less than a unit, in the BAASMTC table.

The "error" of 0.500047 unit in the twelfth decimal place of $d^4 \ln (x!)/dx^4$ at $x = 29.4$ recalls the remarks of J. W. L. GLAISHER as quoted by L. J. COMRIE in N **72**, *MTAC*, v. 2, p. 284–285.

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155.—U. S. COAST AND GEODETIC SURVEY, *Natural Sines and Cosines to Eight Decimal Places*, 1942; see *MTAC*, v. 1, p. 11, 56, 64–65, 87. Now sold by the Superintendent of Documents, Washington at $3.00 (instead of $1.75 in 1942).

A. The Coast and Geodetic Survey has issued the following list of errors in this edition, which, with the errata we have previously published, are all corrected in copies (still dated

1942) now offered for sale:

Page	Function	Angle	Correct Value
44	sin	3°30′45″	2663*0*
45	sin	3°39′24″	7775*6*
108	cos	diff. at top of column	*75*
154	cos	12°44′44″	*3*5944
190	sin	15°43′47″	*0*9981
172, bottom of page, right side 345° sin + should be			345° cos +

B. Mr. E. G. H. Comfort, Illinois Institute of Technology, who drew the above matter to our attention, notes that the 8th decimal of the value for sin 72°21′52″ is one unit too small.

UNPUBLISHED MATHEMATICAL TABLES

76[C].—Clovis Faucher, *Table de Logarithmes à 10 Décimales.* MS. in possession of the author, 33 rue de Bel-Airs, Poitiers, France, xx, 574 p., beautifully written and neatly bound. 20.5 × 31 cm.

In this manuscript, loaned in 1948 for our inspection, Mr. Faucher tells us that he was "Géomètre en chef honoraire; ancien chef des services topographiques de la Côte d'Ivoire, de la Haute Volta et du Soudan français."

The main part of the table is arranged in three columns: (i) Numbers (N) in black; (ii) first five figures of Log N in red; (iii) Logarithmes complémentaires (L.c.) in blue. The argument column is the red column, where the range of values may be said to be 0(.00001).99999, if decimal points are inserted. Corresponding to each of these values the antilogarithm is given in the first column, to 11 digits (rounded off from 13 digit calculation) up to .69999, and to 10 digits (rounded from 12 digits) corresponding to .7(.00001).99999.

In the blue column are the remaining five decimals to be added to the right of the corresponding red-column entry. There are some indications about differences and throughout are attached signs to refine last digits: +(equivalent to .25), −(equivalent to .50), ×(equivalent to .75).

Suppose that it were required to find the logarithm of $\pi \approx 3.141592653$. Then

$$N = 31415\ 92653\times$$
$$n = \mathit{31415\ 21237}- \text{ (next below } N \text{ in table) } \log n = 49714$$
$$N - n = 00000\ 71416+$$

next below *00000 71415*
corresp. Δ 1

Then $\log (N - n) = \bar{5}.85379-$ and L.c. $= \log (N - n) - \log n = \bar{5}.35665-$, corresponding to which in the log column is 98727 whence the required result

$$\log \pi \approx .49714\ 98727.$$

Thus the table is a combination of antilogarithms and of a species of subtraction logarithms.

R. C. A.

77[D].—Ernest Clare Bower (1890–), *Natural Circular Functions for decimals of a circle.* MSS. in possession of author, Douglas Aircraft Company, 3000 Ocean Park Boulevard, Santa Monica, Cal. Listed and punched card copies are available at nominal cost from NBSINA, Univ. California, 405 Hilgard Ave., Los Angeles, Cal., and The Rand Corporation, 1500 Fourth Street, Santa Monica, California.

In F. Callet, *Tables Portatives de Logarithmes*, Paris, 1795, tirage 1819 there is a 15D table of sin x and cos x for $x = 0(0^{\mathrm{r}}.001)0^{\mathrm{r}}.5 = 0(0^{\mathrm{g}}.1)50^{\mathrm{g}} = 0(0^{\mathrm{c}}.00025)0^{\mathrm{c}}.125$. This was

checked by: (1) comparing with ANDOYER's 20D table of these functions for $x = 0(1^g)50^g$; (2) differencing, exposing the error in cos $0^q.114$, *for* 98400 96253 51140, *read* 98400 96256 51140; and (3) extensive spot checking with the aid of Andoyer's series. Subtabulation to 25ths, with an IBM tabulator by my expeditious self-checking method of the Lick Observatory, *Bull.*, v. 17, 1935, p. 65–74, gave 15D values which are subject to an error occasionally somewhat exceeding the usual .5 unit rounding error.

The 10 tables derived from these values, contain sines and cosines, with Δ^2 when significant:

15D, 12D, 10D, 8D, 7D, 6D:	$0(0^c.00001)0^c.125$, 250 p., 12500 cards, each
6D, 5D, 4D:	$0(0^c.0001)0^c.125$, 25 p., 1250 cards, each
4D:	$0(0^c.001)0^c.125$, $2\frac{1}{2}$ p., 125 cards.

The circle is the most practical unit of angular measure in essentially every respect, especially for any computing device—desk computator, punched card machine, etc. It eliminates striking out multiples of 360°, 24^h, 4^q, 6400^m, and $2\pi^r$, and the constant reduction from one unit to another or to a larger unit because the advantage of decimalization is completely realized. The number before the decimal point denotes whole circles, cycles, revolutions, or days, and the decimal is the angle for which functions may be wanted.

E. C. BOWER

EDITORIAL NOTE: The Callet error noted above was corrected in the 1899 *tirage*, and possibly much earlier. There is a copy of the 15D table, for $x = 0(0^c.00001)0^c.125 = 0(0^g.004)50^g$, 250 p., 36.7 × 28 cm., in the Library of Brown University.

78[K].—J. ARTHUR GREENWOOD, *Table of the Double Exponential Distribution*, Ms. in possession of the author, 25 Winthrop St., Brooklyn 25, N. Y.

This table was computed for use in the theory of statistical extreme values. The functions $V(y) = \exp[-e^{-y}]$ and $v(y) = \exp[-y - e^{-y}]$ were introduced by R. A. FISHER & L. H. C. TIPPETT, in Camb. Phil. Soc., *Proc.*, v. 24, 1928, p. 180–190. They were further discussed by E. J. GUMBEL (Institut Henri Poincaré, *Annales*, v. 5, 1935, p. 115–158), who has given (*Annals Math. Statistics*, v. 12, 1941, p. 163–190) a table of $V(y)$ for $y = [-2(.25)+6; 5D]$.

The present table gives $V(y)$ and $v(y)$, for $y = [-3(.1)-2.4(.05)0(.1)4(.2)8(.5)17; 7D]$, with modified second differences.

In addition to its statistical use, this table may be used as an inverse log log table (*MTAC*, Q **4**, v. 1, p. 131; QR **9**, **12**, **30**, **38**, v. 1, p. 336, 373, v. 2, p. 374, v. 3, p. 398). If $y = -x \ln 10 - \ln\ln 10 = \text{approx.} -2.30258\,50930\,x - 0.83403\,24452$, then $V(y) = \text{illolog}\,x$ (in CHAPPELL's notation, *MTAC*, Q **4** note; red lologs must be used in entering CHAPPELL, who gives them with positive mantissae).

J. C. P. MILLER (Camb. Phil. Soc., *Proc.*, v. 36, 1940, p. 286) gives 4S values of $\exp\exp x$, $\exp\exp\exp x$, $\exp\exp\exp\exp x$, for $x = -4(1)+5, -4(1)+3, -4(1)+1$, respectively.

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AUTOMATIC COMPUTING MACHINERY

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TECHNICAL DEVELOPMENTS

Our contribution under this heading, appearing earlier in this issue, is "Piecewise Polynomial Approximation for Large-Scale Digital Calculators," by J. O. HARRISON, JR., & Mrs. HELEN MALONE.

DISCUSSIONS

Conversion of Numbers from Decimal to Binary Form in the EDVAC[1]

All of the electronic high-speed computing machines now under construction or being proposed are able to accept data in some form of the binary-coded decimal notation. The arithmetic units of those machines referred to as "decimal" computers are so designed that they automatically perform the corrections required to yield the results of arithmetic operations in the same form as the original data. Those machines referred to as "binary" computers have arithmetic units which perform true binary addition, multiplication, etc. For the latter, then, it is necessary to translate the original binary-coded decimal number into a true binary number before performing any computations for the problem at hand. Unless auxiliary equipment for this purpose has been provided, this operation must be programed on the computer. It follows, also, that before the final results can be printed, the coder must program the conversion of the binary answers to the binary-coded decimal notation. This paper, however, will present only the conversion of the original data from the decimal to the binary form, as programed for the EDVAC, a binary computer being built at the Moore School of Electrical Engineering, University of Pennsylvania.

The code given is designed for speed in conversion. For any computer, speed can obviously be increased by:

(1) choosing more rapid orders: e.g., using shift instead of multiplication
(2) keeping the number of orders to be executed down to a minimum.

For a computer with a delay-line type of memory having appreciable and variable access times, speed can further be improved by:

(3) choosing the sequence of successive locations for storage of orders and numbers in the memory appropriately.

For a computer having a four-address system of orders, still further speed can be achieved by:

(4) suitably choosing the location (non-consecutive in general) of successive instructions (fourth-address).

Most of the above is accomplished at the expense of memory space. This is not a drawback, since it is intended that the conversion will be performed when the EDVAC is not simultaneously carrying out any other program. In view of this, although it is possible to set up subiterations and thus reduce the number of order-words and produce a more compact routine, this was deliberately not done where it interfered with obtaining maximum speed.

It is assumed that each ten-digit decimal number, N, is introduced into the EDVAC in binary-coded decimal form with each decimal digit represented by a four-digit binary equivalent (e.g., 1001 for the decimal digit 9, 1000 for 8, etc.), the four binary-digit groups being in the same sequence as the original decimal digits. The sign is recorded in the correct binary sign position as if the number were binary, while the three least significant binary digits are recorded as zeros, making a total of 43 binary digits and a sign. It is further assumed that the decimal point is, in all cases, located immediately to the left of the most significant digit. The unsigned part of the binary-coded decimal number is, therefore, introduced into the computer in the form:

$$N_d = 2^{-4}n_1 + 2^{-8}n_2 + 2^{-12}n_3 + \cdots + 2^{-40}n_{10},$$

where n_i = the ith binary-coded decimal digit. For the purpose of this paper, it is sufficient to consider the unsigned part of the number since the sign and the last three binary zeros do not affect the method of computation.

The true binary representation of the magnitude of the number, N, is

$$N_b = 10^{-1}n_1 + 10^{-2}n_2 + 10^{-3}n_3 + \cdots + 10^{-10}n_{10}.$$

Comparison of the form of the two numbers N_d and N_b indicates one direct method of conversion, as follows: 1) Extract the binary-coded decimal digit n_{10}. 2) Multiply n_{10}

by the binary equivalent of 10^{-1}. 3) Extract the binary-coded decimal digit n_9. 4) Obtain $n_9 + 10^{-1}n_{10}$. 5) Multiply by 10^{-1} to obtain $10^{-1}n_9 + 10^{-2}n_{10}$, etc.

Each of the steps enumerated above, with the exceptions of the extract operations, can be obtained by the execution of just one command. In order to extract a group of digits contained within a word, with the commands available in the EDVAC, it is necessary to use two shift operations.

A more efficient procedure is to utilize just one shift operation and to derive coefficients (instead of the constant 10^{-1} employed above) that will compensate for the extraneous information introduced by permitting the digits less significant than the required n_i to remain in the computation. Let $a_1, a_2, a_3, \cdots, a_{10}$ equal the desired coefficients. Then, following the procedure outlined above, we obtain in succession:

$$1)\ 2^{-4}n_{10},\quad 2)\ a_{10}(2^{-4}n_{10}),\quad 3)\ 2^{-4}n_9 + 2^{-8}n_{10},\quad 4)\ 2^{-4}n_9 + 2^{-8}n_{10} + a_{10}(2^{-4}n_{10}),$$
$$5)\ a_9[2^{-4}n_9 + 2^{-8}n_{10} + a_{10}(2^{-4}n_{10})],\quad 6)\ 2^{-4}n_8 + 2^{-8}n_9 + 2^{-12}n_{10},$$
$$7)\ 2^{-4}n_8 + 2^{-8}n_9 + 2^{-12}n_{10} + a_9[2^{-4}n_9 + 2^{-8}n_{10} + a_{10}(2^{-4}n_{10})],$$
$$8)\ a_8\{2^{-4}n_8 + 2^{-8}n_9 + 2^{-12}n_{10} + a_9[2^{-4}n_9 + 2^{-8}n_{10} + a_{10}(2^{-4}n_{10})]\}.$$

Continuing the above process, we obtain at the 29th step the number (A),

$$\begin{aligned}(A) = a_1(2^{-4}n_1 + 2^{-8}n_2 + \cdots + 2^{-40}n_{10} + a_2(2^{-4}n_2 + 2^{-8}n_3 + \cdots + 2^{-36}n_{10} \\ + a_3(2^{-4}n_3 + 2^{-8}n_4 + \cdots + 2^{-32}n_{10} + \cdots + a_9(2^{-4}n_9 + 2^{-8}n_{10} + a_{10}(2^{-4}n_{10}))\cdots)\end{aligned}$$

Combining the coefficients of n_i, we readily put (A) in the form:

$$\begin{aligned}(A) = {} & 2^{-4}a_1n_1 + 2^{-4}a_1(2^{-4} + a_2)n_2 + 2^{-4}a_1(2^{-8} + a_2(2^{-4} + a_3))n_3 \\ & + 2^{-4}a_1(2^{-12} + a_2(2^{-8} + 2^{-4}a_3 + a_3a_4))n_4 \\ & + 2^{-4}a_1(2^{-16} + a_2(2^{-12} + 2^{-8}a_3 + 2^{-4}a_3a_4 + a_3a_4a_5))n_5 \\ & + \cdots + 2^{-4}a_1(2^{-36} + a_2(2^{-32} + 2^{-28}a_3 + 2^{-24}a_3a_4 \\ & + \cdots + 2^{-4}a_3a_4a_5 \cdots a_9 + a_3a_4a_5 \cdots a_{10}))n_{10}.\end{aligned}$$

Equating the coefficients of n_i in (A) with those required for the converted binary number N_b, we get ten equations for the determination of the ten constants $a_1, a_2, \cdots, a_{10}$:

$$2^{-4}a_1 = 10^{-1},\quad 2^{-4}a_1(2^{-4} + a_2) = 10^{-2},\quad \text{or}\quad 2^{-4} + a_2 = 10^{-1},$$
$$2^{-8} + a_2(2^{-4} + a_3) = 10^{-2},\quad 2^{-12} + a_2(2^{-8} + 2^{-4}a_3 + a_3a_4) = 10^{-3} \cdots$$
$$2^{-36} + a_2(2^{-32} + 2^{-28}a_3 + 2^{-24}a_3a_4 + \cdots + a_3a_4 \cdots a_{10}) = 10^{-9}.$$

It is apparent that the following is a solution of these equations:

$$a_1 = 8/5,\quad a_2 = 3/80,\quad a_3 = a_4 = \cdots = a_{10} = 10^{-1}.$$

The EDVAC operates with a fixed binary point located in front of the most significant digit of the number and is therefore incapable of storing numbers ≥ 1. It is evident that a_1 exceeds the capacity of the machine. Therefore the routine divides by the reciprocal of a_1 thus keeping all of the numbers within the bounds of the computer.

For those less familiar with the EDVAC, the following details will assist in interpreting the code given:

Data are read into and out of the machine by means of magnetic wires, three of which are used in this routine. The memory of the EDVAC consists of 128 acoustic delay lines, each having 8 words, thus giving a total internal memory capacity of 1024 words. Since this computer operates in the true binary system, orders are written in the octal notation. The word length of both numbers and orders in this machine is 44 binary characters. There is a space of four pulse positions between words. Since the pulse repetition rate is equal to one megacycle, the time it takes to read one word (a minor cycle) is 48 microseconds. A number is represented as 43 binary digits plus a sign—the sign occupies the 44th binary position. Each order word consists of four addresses of ten characters each and an instruction code of four characters. For most instructions, the information is distributed in the following manner:

Address no. 1 (P_1–P_{10})[2]	Address no. 2 (P_{11}–P_{20})[2]	Address no. 3 (P_{21}–P_{30})[2]	Address no. 4 (P_{31}–P_{40})[2]	Operation (P_{41}–P_{44})[2]
Memory position from which 1st operand is selected	Memory position from which 2nd operand is selected	Memory position to which result of operation is sent	Memory position at which next order stands	Instruction Code

Instruction Codes used in the conversion of binary-coded decimal numbers to the binary notation:
Note: (M) means contents of memory position M.

	Address no. 1	Address no. 2	Address no. 3	Address no. 4	Operation[3]
Addition					
Obtain (X) + (Y) and store the sum in Z.	X	Y	Z	next order	A
Subtraction					
Obtain (X) − (Y) and store the difference in Z.	X	Y	Z	next order	S
Multiplication with round-off					
Obtain the rounded product of (X) and (Y) and store in Z.	X	Y	Z	next order	M
Division with round-off					
Obtain the rounded quotient of (X) ÷ (Y) and store the result in Z. Here (X) must be less than (Y).	X	Y	Z	next order	D
Comparison					
Compare (X) with (Y); if (X) ⩾ (Y), the next order is contained in G; if (X) < (Y), the next order is contained in L.	X	Y	L	G	C
Extraction					
a) Shift (X) n places to the left (n is written as two octal digits), replace the digits in the *first* address of (Y) with the corresponding digits of the shifted (X), and store in Y.	X	0 n 1	Y	next order	E
b) Same as (a) above, except that (X) is shifted to the right.	X	1 n 1	Y	next order	E
c) Shift (X) n places to the left, replace the digits of the third address of (Y) with the corresponding digits of the shifted (X), and store in Y.	X	0 n 3	Y	next order	E
d) Same as (c), except that (X) is shifted to the right.	X	1 n 3	Y	next order	E
e) Shift (X), exclusive of the sign, n places to the left, replace (Y) by the shifted value of (X) with original sign of (X).	X	0 n 7	Y	next order	E
f) Same as (e), except that (X) is shifted to the right.	X	1 n 7	Y	next order	E
Wire Orders					
a) Write on wire n, (n is written as two octal digits) starting with memory position X through Y in sequence.	X	0 1 n	Y	next order	W
b) Read from wire n, read the first word into X, the next word into X + 1, etc., in sequence through Y.	X	0 2 n	Y	next order	W
c) Read from tape n, store the words into the memory address specified by the fifth address of each word until the fifth address = Y, store that word in Y and continue to the next order. X has no significance in this order.	X	0 3 n	Y	next order	W

The following program covers conversion of Input Data on Wire no. 2 in decimal form to equivalent data, similarly arranged, but in binary form on Wire no. 3. Any number of words from 1 to the full capacity of the wire may be converted at one time. The number of words desired is expressed in the form $N = 2^9 n + r$ with n integral and $0 < r \leq 2^9$. The values of n and r are set up respectively in octal form on the "Address no. 1" and "Address no. 3" positions of the "Auxiliary Input" switches of the EDVAC. All the remaining switches are set at zero. These data are read into position 1001 of the internal memory by suitable use of the "Special Order" switches of the machine.

The coding below is typed on the Input Typewriter to form a corresponding wire if one is not already available in the library of routines. This wire is mounted on the EDVAC Drive no. 1, the wire carrying the decimal data on Drive no. 2, and a blank wire (or one carrying data no longer needed) on Drive no. 3. The "Special Order" switches are now set for the order:

W 0000 0301 1042 0000

and the "Initiate" button is again pressed. This will cause the code to be read off Wire no. 1 into the internal memory. When the EDVAC has halted again as indicated by the blue pilot, the operator sets the "Mode of Operation" dial to "Normal-to-Completion" and presses the "Initiate" button once again. This will begin the conversion process and produce binary data on Wire no. 3 equivalent to the decimal data on Wire no. 2 in the corresponding positions, continuing automatically until the number of words specified have been converted, when the blue pilot will light again to indicate completion of the job.

The initial orders of the routine read a group of words from Wire no. 2 into the memory. The first group consists of r words and is stored in memory positions $1, 2, 3, \cdots, r$. Succeeding groups will consist of 512 words and will be stored in positions $1, 2, 3, \cdots$ 777 (octal notation). The next 4 instructions modify the subsequent commands for the purpose of storing converted numbers, testing each iteration to determine when converted data are to be read out, testing for the end of the program, etc.

The coding for the actual conversion from the decimal to the binary notation proceeds as outlined above using the coefficients $a_1 = 8/5$, $a_2 = 3/80$, $a_3 = a_4 = \cdots = a_{10} = 10^{-1}$. The successive stages of each iteration are coded below in detail in order to save the time required to execute the modifications necessary when a sub-iteration is used. As previously stated, the various addresses have been chosen for optimum speed in execution rather than for compactness. The extreme possible error in conversion by the process given is, closely, $3/2 \times 10^{-13} = 4/3 \times 2^{-43}$.

The speed of conversion is approximately 10 words per second, based on an input-output speed of 30 words per second. If and when the input-output is replaced by more efficient magnetic tape devices, the economies effected in this routine will become more apparent. For example, when an input and output speed of 400 words per second is attained, the rate of conversion will be 27 words per second.

Program for Converting Binary-Coded Decimal Numbers to the True Binary Notation

Time to Perform Operation in Minor Cycles	Memory Position of Order (5th Address)	Operation	Address no. 1	Address no. 2	Address no. 3	Address no. 4
11	0000	E	1001	0003	1003	1003
**	1003	W	0001	0202	0000	1002
5	1002	E	1003	0241	1006	1007
7	1007	E	1003	0003	1055	1006
10*	1006	E	0000	0007	1054	1005
4	1005	E	1006	1243	1050	1011
11	1011	E	1054	0447	1047	1004
47	1004	M	1016	1047	1052	1013
5	1013	E	1054	0407	1056	1010
10	1010	A	1052	1056	1047	1012
49	1012	M	1016	1047	1052	1023

Time to Perform Operation in Minor Cycles	Memory Position of Order (5th Address)	Operation	Address no. 1	Address no. 2	Address no. 3	Address no. 4
4	1023	E	1054	0347	1056	1017
13	1017	A	1052	1056	1047	1014
47	1014	M	1016	1047	1052	1033
5	1033	E	1054	0307	1056	1020
13	1020	A	1052	1056	1047	1015
46	1015	M	1016	1047	1052	1043
6	1043	E	1054	0247	1056	1021
9	1021	A	1052	1056	1047	1022
49	1022	M	1016	1047	1052	1053
4	1053	E	1054	0207	1056	1027
13	1027	A	1052	1056	1047	1024
47	1024	M	1016	1047	1052	1063
5	1063	E	1054	0147	1056	1030
13	1030	A	1052	1056	1047	1025
46	1025	M	1016	1047	1052	1073
6	1073	E	1054	0107	1056	1031
9	1031	A	1052	1056	1047	1032
49	1032	M	1016	1047	1052	1103
6	1103	E	1054	0047	1056	1041
8	1041	A	1052	1056	1047	1051
50	1051	M	1026	1047	1052	1113
13	1113	A	1054	1052	1044	1050
57*	1050	D	1044	1035	0000	1045
10	1045	S	1006	1042	1006	1037
14/15	1037	C	1006	1042	1055	1006
**	1055	W	0001	0103	0000	1040
4/6	1040	C	1001	1042	1034	1036
16	1036	S	1001	1042	1001	1046
13	1046	E	1042	1133	1003	1003
	1034	H[4]	0000	0000	0000	0000

Storage of Constants:

1016	+014 6314 6314 6315	(=1/10)
1026	+004 6314 6314 6315	(=3/80)
1035	+120 0000 0000 0000	(=5/8)
1042	+000 1000 0000 0000	($=2^{-10}$)

FLORENCE KOONS & SAMUEL LUBKIN

NBS

[1] The basic idea for the conversion method discussed herein is due to Dr. LUBKIN.

[2] Position (P_1–P_{10}), etc., are the 10 most significant binary digits of the order word.

[3] Letters are used instead of the binary notation for easier association.

[4] H signifies Halt Order.

* Average for n large. Minimum and maximum may deviate from this by ±7 minor cycles.

** Time required to execute this operation is essentially time required to move input and output devices.

BIBLIOGRAPHY Z–VII

1. LEON BRILLOUIN, "Les grandes machines mathématiques Américaines," *Atomes*, no. 21, Dec. 1947, p. 400–404, illustrs. 20.3 × 26.7 cm. This paper is preceded by an introductory article entitled, "L'evolution des machines à calculer," by JEAN PÉRÈS, p. 399–400. Both articles appear under the title "Les cerveaux électroniques."

2. W. J. Eckert, "Electrons and computation," *Sci. Mo.*, v. 68, 1948, p. 315–323. 19 × 26 cm.

A description of the design and operation of the new IBM Selective Sequence Electronic Calculator; see *MTAC*, v. 3, p. 216–217, 326 (**6**).

3. Harrison W. Fuller, "Numeroscope for cathode-ray printing," *Electronics*, v. 21, 1948, p. 98–102, illustrs. 27.9 × 20.3 cm.

The Numeroscope, a rapid large-scale computer printer, is an electronic device for tracing upon the screen of a cathode-ray tube the patterns of the Arabic numerals from one to zero. This makes it possible to build a printer that will display the result of a computation upon an assembly of cathode-ray tubes and that will record the displayed quantity on fast film. The article discusses many of the circuit techniques used in the design of the Numeroscope.

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4. Harvard University, Computation Laboratory, *Annals*, v. 16: *Proceedings of a Symposium on Large-Scale Digital Calculating Machinery*, Jointly Sponsored by the Navy Department Bureau of Ordnance and Harvard University at the Computation Laboratory, 7–10 January 1947. Cambridge, Mass., Harvard Univ. Press, 1948, xxix, 302 p. 19.7 × 26.7 cm. $10.00.

This handsome, finely illustrated volume is a detailed report of the Symposium of which we have already published the program, and a list of members registered, in *MTAC*, v. 2, p. 229–238. Other details are now presented in the volume under review, but especially the texts of the 31 scientific papers delivered. We shall now endeavor to suggest the noteworthy contents of these papers.

Mr. R. H. Babbage presented an interesting account of some of the difficulties which Charles Babbage met and overcame in his pioneering efforts in the field of mechanical computation.

Babbage invented two calculating machines: the difference engine (financed by the British Government) and the analytical engine. He was handicapped by the fact that he not only had to design the machine parts but was even forced to design tools for fabricating them. In his notes he states that "some of the most enlightened employers and constructors of machinery, who have themselves contributed to its advance, have expressed to me their opinion that if the calculating engine itself should entirely fail, the money expended by Government in the attempt to make it would be well repaid by the advancement it had caused in the art of mechanical construction."

The construction of the difference engine was carried on during many years, but due to circumstances beyond Babbage's control the engine was not completed. The inventor turned to work on his second invention, a more powerful calculating machine called the analytical engine, which he expected to execute not only such work as the difference engine had been planned to perform, but every kind of analytical operation indicated by formulae. However, he encountered serious obstacles in construction of the analytical engine. The difficulties were not so much in the design of the engine as in its construction. It is interesting to speculate on what might be the present state of the art of numerical computation by large-scale automatic machinery had Babbage been able to accomplish the construction of his analytical engine and put it into active service.

The Mark I, described by Mr. R. M. Bloch and located at the Harvard Computation Laboratory, is an electro-mechanical device, having an operational speed of 200 cycles per minute, and controlled by coded instructions on a teletype paper tape. The calculator has storage capacity of seventy-two 23-digit numbers, together with their signs. It has three types of input devices: 60 "constant" registers (manually-set 10-pole valve switches), used for storage of numerical constants, tolerances, increments, parametric

values, etc.; two standard IBM card feeds, serving not only to permit the introduction of quantities into the machine from punched cards but also as an extension of the storage unit; and three interpolator mechanisms used for the finding and reading of functional values stored in coded form on "value" tapes. The machine has two output devices: IBM electromatic typewriters and IBM card punches. Plug boards are provided for controlling the typing so that the output of the machine may be arranged in the exact form desired for publication by photo-photography. The machine performs the basic arithmetical operations of addition, subtraction, multiplication, and division. Various applications of the machine which had been made were mentioned.

The ENIAC (Electronic Numerical Integrator and Computer) which was designed and built in the Moore School, under the sponsorship of the Ordnance Department of the U. S. Army, was described in detail by Dr. L. P. TABOR.

The ENIAC operates on ten-decimal-digit numbers and performs its calculation by counting voltage pulses, which are formed in a cycling unit in groups of 1, 2, 4, 9, and 10. The basic frequency of pulse generation is 100,000 per second. The fundamental cycle of the computer is the addition time, which is 200 microseconds. The remaining built-in operations are performed exceedingly fast: 300 multiplications, 50 or more divisions or square root extractions in a second.

Since an article on the ENIAC has appeared in this journal (see *MTAC*, v. 2, p. 97–110), further details here are unnecessary.

Next, Dr. S. B. WILLIAMS briefly traced the development of the BTL relay computers from the time of the suggestion by Dr. G. R. STIBITZ regarding the use of telephone relays and teletype apparatus for numerical computation through the design of the large-scale Relay Computing System. After mention of the first application of Stibitz's ideas, to the "complex computer," which added, subtracted, multiplied and divided complex numbers, operating from a keyboard and printing results on a teletype printer, and with a preliminary discussion of the "excess-three" representation of decimal digits and the bi-quinary representation of digital values, he explained thoroughly the design features of the Bell Telephone Laboratories' Relay Computing System.[1] This computing system is among the most flexible and reliable of existing large-scale computing systems, and has many novel and interesting features. By the use of diagrams and slides with his talk, Dr. Williams made clear the manner in which this computing system functions.

In his talk, Mr. R. V. D. CAMPBELL described the Dahlgren Calculator—Mark II—which was then under construction at the Harvard Computation Laboratory for the Bureau of Ordnance of the United States Navy.[2] This machine, of ten decimal-digit capacity, differs from the Mark I among other features in that the arithmetic operations are performed by electro-mechanical relays, not mechanical counters, and manually-set dial switches are used only in a subsidiary capacity. Furthermore it has an increased internal storage and uses a "floating decimal point." An important feature of the calculator is its ability to operate as a unit on one problem or its possible use as two independent halves, each half operating on a separate problem.

A number N, with floating-decimal point, is expressed in the Mark II in terms of another number p, and an integer j, according to the representation

$$N = p \times 10^j, \quad 1 \leqslant p < 10, \quad \text{and} \quad -15 \leqslant j \leqslant +15.$$

A storage register is an assemblage of 62 relays, 16 of which are used for routing of quantities into and out of the register and 46 of which are necessary for the representation of N (one relay for the algebraic sign, 5 relays for j, and 4 relays for each of the ten decimal digits of p). The machine automatically adjusts exponents, j, in addition and multiplication. It has two adders and four multipliers. Subtraction is accomplished by use of complements on nines in the adder; multiplication is compounded from the first five multiples of the multiplicand and algebraic addition.

Permanently available within the calculator are the reciprocal, the reciprocal square root, the logarithm, the exponential, the cosine, and the arctangent. In addition, the machine contains four input devices which supply it with coded tables of functions punched in paper tapes.

Four input mechanisms are available for introducing into the calculator the orders contained in sequence tapes. Also provided are four input devices for introducing quantities into the machine. The output devices consist of four automatic typewriters.

The Mark II contains about 13,000 electro-mechanical relays. These relays, designed especially for the machine, operate in from six to ten milliseconds. About one-third of the relays are of the "latch" type: they can be locked mechanically in position. The storage relays in the internal memory are of this type and thus maintain their position in the event of power failure.

The machine is of the synchronous type having a basic cycle of one second duration. Thirty orders are executed in each cycle, by the machine operating as a whole or by each of the two halves in split operation. Programed checking of machine operations is used. In addition, the machine sounds an alarm in case a number read into one of its units exceeds the capacity of the unit.

The theme of Dr. A. W. Wundheiler, in his paper on "Problems of mathematical analysis involved in machine computations" was that "pure numerical computation" cannot replace mathematical analysis. "A bare numerical result without statement of the associated error has no scientific value, and only an analysis in general terms can provide an expression for the associated error." Dr. Wundheiler gave a broad survey of urgent problems in error analysis, stressing the limitation of the "common sense" approach to questions of convergence and illustrating by use of slowly convergent series, the Gibbs phenomena in Fourier's series, and successive approximation formulae yielding non-convergent results. He pointed out the difficulty in determining how fine to make the meshes used in connection with the application of finite-difference techniques to approximate solutions of boundary value problems. Also, the sums of round-off and truncation errors are not independent, as a consequence of which the accuracy available with a given approximation method and a given number of digits may be severely limited.

Perhaps the most urgent need for further development of error analysis, according to Dr. Wundheiler, lies in the field of second-order partial differential equations. A second need is for further analysis of round-off error.

In his talk on "The organization of large-scale calculating machinery" Dr. Stibitz discussed the "drive" behind numerical computation, namely, the fact that computing is done because the computed results are useful and frequently necessary for the attainment of an important end. Treating the role of large-scale computing machines in important and extensive work programs, from the standpoint of economy and also of effect on the structure of the work group, he discussed external organization of such machines as being influenced by and also tending to mold their environment. In connection with external environment, Dr. Stibitz discussed machine flexibility, repetitive computation versus single problem operation, machine vocabulary, the interpretation of mathematical symbols and instructions by the machine, and diagnostic equipment.

In his discussion of the internal organization of large-scale computing machinery, Dr. Stibitz touched on fixed-cycle and variable-cycle designs, complexity of the control mechanism and flexibility. He pointed out that the internal organization is essentially under the control of the designer, subject to the limitation that interference with requirements of the external organization must be avoided. He proposed a control level of intelligence lying between the control tape and the arithmetic unit, at which level would occur translation from mathematical vocabulary to machine language, and also, the interpretation of instruction, concerning printing and other auxiliary matters.

In his paper on "Mercury delay lines as a memory unit," Dr. T. K. Sharpless described a dynamic-type storage unit for electronic digital computers. If provisions are made for recirculating a pulse train repeatedly through a delay device with sufficient control over attenuation and distortion, the device medium serves as a practicable memory unit for electronic computers.

The transmissive losses for acoustic waves in a mercury column are small; a mercury column together with piezo-crystal input and output provides, therefore, a good delay medium for electrical pulses. Delay variation with temperature limits the length of the mercury

column that it is possible to use. A feasible memory system would consist of a bank of mercury delay lines with associated recirculation circuitry, so mounted as to minimize thermal potential differences, and with one line of the bank controlling the frequency of the pulses by means of an automatic frequency control unit, so as to keep constant the storage capacity of each line.

Discussing "Slow electromagnetic waves," Professor L. N. Brillouin stated that the delay of a short pulse, of the type used to represent binary digits in electronic computers requires: (a) a very broad passing band, (b) inclusion of low frequencies, preferably (to avoid the complication of transmission as a modulated carrier), (c) no distortion, hence constant velocity of propagation and constant attenuation for all frequencies passed.

Wave guides, spiral delay lines and lumped artificial lines were discussed as sources of production of slow electromagnetic waves in relation to requirements (a), (b) and (c).

In the case of wave guides, it appears that phase velocity and attenuation depend strongly upon frequency and, therefore, requirement (c) is not met.

The spiral-delay-line method for producing slow waves is a practical one. This type of line yields wide bands with very small phase distortion, but with a certain amount of amplitude distortion.

Filter theory is applicable in the case of lumped artificial lines. The velocity of propagation in a standard low-pass filter does not remain constant throughout the passing band. However introduction of a certain amount of mutual inductance between sections of the filter both improves the delay characteristics and maintains the velocity of propagation constant over a wide range of frequencies.

Included in the printed copy of Professor Brillouin's talk are two appendices giving mathematical analysis of propagation along a solenoid and of a low-pass filter with mutual inductance.

Dr. J. W. Forrester's paper, "High-speed electrostatic storage," reviewed requirements which an electrostatic storage device must meet for application to high-speed electronic computation as contemplated in a research program of the Servomechanisms Laboratory sponsored by the Special Devices Division of the Office of Naval Research. Emphasis was being placed on (1) high signal-to-noise ratio; (2) indefinite storage time, no restriction being imposed on the order or number of times a storage position is used; (3) ready accessibility to storage (6 microseconds allotted to storage control and operation); and (4) simple and reliable mechanical and design characteristics.

Dr. Forrester gave a qualitative treatment of a beam-deflection electrostatic storage tube, intended for parallel operation in banks of as many tubes as there are binary digits in the stored numbers, and operating by the utilization of well-known secondary-emission effects. Outstanding problems in the development of the tube were discussed, and the results of experimental work of the Servomechanisms Laboratory were treated briefly.

Dynamic optical and static magnetic storage were discussed by Dr. B. L. Moore. The optical system consisted of a rotating disc coated with a phosphorescent material, a modulated light source on one side of the drum, a photo-electric cell on the diametrically opposite side, and an erasing and feedback arrangement between the receiving cell and the light source, in the direction of rotation of the drum. The storage device would function much the same as a mercury-delay line storage device with light spots on the face of its rotating drum playing a role similar to that of the acoustic wave trains in the former. Difficulty had been experienced with the erasing and feed-back arrangement.

The well-known magnetic drum storage was described as a more promising device than the phosphor drum or disc. One big advantage of the magnetic-storage is its static feature: power failures do not cause loss of information stored on the magnetic material.

Dr. Jan Rajchman, summarizing the specifications for an ideal memory organ for a digital computing machine, stated "The memory organ should be able to register in as short a writing time as possible any selected one of as many as possible on-off signals and should be able to deliver unequivocally the result of this registration after an arbitrarily long or short time, with the smallest possible delay following the reading call."

The selectron, designed in an attempt to meet these ideal requirements, appeared to

show promise as a memory unit for electronic digital computers. Dr. Rajchman explained very clearly the design and operation of this tube, illustrating his talk effectively by the use of slides. The interested reader will find a complete description of the selectron in an article by Dr. Rajchman in *MTAC*, v. 2, p. 359–361.

Dr. A. W. TYLER discussed the characteristics of photographic emulsions and phosphorescent materials, from the standpoint of their use to provide permanent and supplemental storage for electronic digital computing machinery. He described the handling of photographic film, high-speed scanning techniques, and the like.

He concluded that photographic film would be a useful form of permanent storage for use with electronic computers, and that phosphor-coated film showed promise of developing into a satisfactory supplemental storage medium.

In his paper, "Method of finite differences for the solution of partial differential equations," Prof. R. COURANT briefly indicated some directions in which theoretical mathematical efforts must be turned if new scientific results are to come from the development of high-speed automatic computing machines. He was concerned more specifically with the field of boundary and initial-value problems of partial differential equations of physics. He discussed numerical methods as replacing an analytical problem, P, by an approximate problem P_h, depending on a parameter h, and having a solution S_h obtainable by computational methods. The main question is: "for small values of h, when P_h approximates P, is S_h likewise an approximation to the desired solution S of P"?

Dr. Courant discussed methods for increasing the speed of convergence of S_h toward S as h approaches zero, for example, by increasing the order of equations, by use of general nets for finite difference schemes, and the like. He mentioned that numerically following the development and propagation of discontinuities (shocks) will probably require extensive theoretical and numerical procedure. The relevance of theoretical questions of existence and uniqueness for an understanding of physical problems was emphasized.

Dr. R. J. SEEGER's remarks concerned computational techniques applicable to problems in the field of explosive phenomena. Problems concerning the thermal sensitivity of an explosive and underwater explosion phenomena were discussed.

The complexity of the problems is sufficiently illustrated by the exhibition of one of the differential equations involved, say the non-dimensional form of the equation involved in the linear case of the first problem:

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\xi^2} + e^{-1/\theta}.$$

Various methods which have been devised for the approximate solution of this and the remaining differential equations were discussed. Approximate solutions obtained by the Naval Ordnance Laboratory, the Harvard Computation Laboratory, the Mathematical Tables Project (now a part of the NBS Computation Laboratory), and the Ballistic Research Laboratory, Aberdeen Proving Ground, were analyzed and discussed in detail.

Typical problems in the field of industrial relationships were formulated mathematically by Prof. W. W. LEONTIEF. The corresponding mathematical problems consisted of solving large systems of linear algebraic equations. Since the most meaningful equations involve forty, a hundred or even more unknowns, the computational task arising in connection with their solution is a formidable one.

An efficient iterative procedure for the solution of large linear systems was mentioned. If the system is

$$[M]\{x\} = \{y^{(0)}\},$$

where $[M]$ is a square matrix of nth degree, possessing an inverse $[M]^{-1}$, and $\{x\}$ and $\{y^{(0)}\}$ are column matrices of n elements each, the solution $\{x\}$ can be obtained as

$$\lim_{n\to\infty} [\{y^{(0)}\} + \{y^{(1)}\} + \cdots + \{y^{(n)}\}],$$

where $\{y^{(k)}\} = [I - M]\{y^{(k-1)}\}$, for $k \geqq 1$, provided the series $\sum_{k=0}^{\infty} \{y^{(k)}\}$ converges. A suffi-

cient condition for convergence is the tending toward zero of all elements of the matrix $[I - M]^k$ as k tends toward infinity.

Prof. H. A. RADEMACHER's remarks concerned truncation and round-off errors, particularly those affecting the accuracy of the numerical integration of ordinary differential equations. It was assumed that numbers were rounded off to k digits in the computations and the accumulation associated with Heun's approximation of first-order linear differential equations was analyzed. The total truncation and probable round-off errors were obtained by use of systems of adjoint differential equations and adjoint difference equations respectively. The orders of the total accumulated errors of the two types were shown to be $(\Delta t)^2$ and $(\Delta t)^{-\frac{1}{2}}$, a fact which holds true for the method independent of the order or number of the equations. It clearly follows that the assumption frequently made by computers using finite-difference methods that accuracy increases as the size of the interval decreases should not be followed blindly.

The Navier-Stokes equations of flow were discussed by Prof. H. W. EMMONS. Many special analytical solutions of these equations are known, but the speaker believed that no analytical solution of the finite oscillations of turbulent flow had been attempted. He proposed a direct numerical attack on the turbulence problem by the use of the "bigger, better, faster, and more reliable computing machinery" that would become available. One might begin by assuming the fluid flowing initially in Poiseuille flow and try to compute the flow characteristics for later times. At sufficiently high Reynolds numbers the rounding errors should grow in magnitude until the computed flow exhibits the characteristics of turbulence. If a correct numerical procedure were used, Dr. Emmons felt it could be used to investigate the adequacy of the continuous treatment of fluid mechanics via the Navier-Stokes equations.

In his talk on "Firing tables," Dr. L. S. DEDERICK described a finite-difference approximation method similar to, but appearing to be substantially superior to, the Heun method. For the equation

$$\frac{dy}{dx} = F(x, y),$$

the typical step taken in accordance with the method treated is illustrated by

$$y_1 = y_0 + hy_0', \qquad y_1 = F(x_0 + h, y_1), \qquad y_2 = y_0 + 2hy_1'.$$

It had been estimated that the total accumulated error associated with this method was appreciably less than that inherent in the Heun method. Dr. Dederick next considered the approximate solution by numerical methods of the trajectory equations and closed his talk with general remarks on ways of programing the ENIAC to exploit its speed in the computation of firing tables.

Concerning the EDVAC type, Dr. J. W. MAUCHLY discussed its extensive internal memory, its minimum of elementary instructions, and ability to store instructions in the internal memory and to modify instructions as directed by other instructions. He also included a discussion of problem preparation on these computers. Furthermore serial operation, the use of "flow-charts," the use of sub-routines, and the preparation of instruction tapes by use of the computer were treated.

Mr. J. O. HARRISON, JR., in his paper on "The preparation of problems for large-scale calculating machinery," placed emphasis on the analysis which must be performed preliminary to setting up the Mark I computer for a problem. He mentioned four steps: (1) decision upon the exact method of computation, (2) selection of a method of checking, (3) determination of the magnitude of intermediate results, and (4) analysis of error. He discussed mathematical checking by differencing, checking identities and repetition of operations on different equipment within the computer. The design of a sequence tape and setting up the Mark I were also treated in the talk.

The general theme of the next seven papers was "Input and output devices."

In a paper on "Application of printing telegraph techniques to large-scale calculating machinery" Mr. F. G. MILLER described the Western Union teletype equipment used, with appropriate modification, in the Mark II computer system. Included in this equipment were page printers, reperforators, transmitters, distributor-transmitters, and the like.

The talk was well illustrated by slides. Credit was given to Messrs. R. F. DIRKES and A. E. FROST, engineers of the Western Union Telegraph Company, for their assistance in the performance of the engineering work involved in the application of the teletype equipment in the Mark II.

In the second talk. Mr. OTTO KORNEI described perpendicular and longitudinal magnetic recording—the terms applying to direction of magnetization in the recording magnetic medium. Subjection of the recording medium to a bias field during erasing to obtain a linear transfer characteristic was explained together with an analysis of the design characteristics of recording and reproducing magnetic heads. The relation between high frequency response and the ratio of coercive force to remanence of a magnetic material was treated graphically, by consideration, based on the hysteresis loop. High absolute values of these two quantities are desirable: remanence to produce high absolute reproducing level, coercive force to offer resistance to accidental demagnetization. The effect of thickness of the recording medium or response was discussed, with the use of experimental data. Gap-width effect was analyzed. The talk ended with a discussion of various ways of producing commercial magnetic recording media.

The numeroscope described by Mr. H. W. FULLER was developed at the Harvard Computation Laboratory for the purpose of high-speed printing. The device consists of cathode-ray tubes together with deflection voltage circuits which generate voltage patterns for the tracing upon the screen of any one of the decimal digits 0 through 9. Several methods for generating the deflection plate voltages for the tracing of a given digit on the tube end were outlined.

Mr. S. N. ALEXANDER discussed the development work at the National Bureau of Standards on the modification of Teletype Corporation equipment to form an input system for electronic computers.

In one such system two initial teletype tapes are prepared independently from the same manuscript. Errors arising either from operator's mistakes or from the equipment are detected by automatic electrical comparison of the tapes. The magnetic input tape for the electronic computer is in this case prepared automatically from one of the corrected initial tapes.

An alternative tape preparation device consists of a system for preparation of a teletype tape from the original manuscript, and independent preparation from the same manuscript of a magnetic input wire or tape with checking against the paper tape and page printing of the instruction sequence. Design details of the two systems were discussed, with the use of slides.

In Dr. MORRIS RUBINOFF's talk the properties of multiple gates were emphasized and input devices of design based on the operation of multiple-gate tubes were described as an example of the application of such tubes. The multiple-gate tube is a multi-grid vacuum tube functioning with several grids used as gating grids. The tube responds to input signals only if all gate grids have been placed at "normal" voltage levels. The tube functions for the passage of voltage pulses, as a multiple-control switch. The circuit schematics for an input device using such gates was described.

Mr. R. D. O'NEAL described a photographic film input reader.[3] It was stated that as many as 50 channels of information could be stored in readable form, on 35 mm. photographic film. The optical system of the reader was described, and checking methods were explained. Parallel operation of an input reader with a high-speed electronic computing machine was discussed. It was stated that most of the experience required for the construction of an input reader of the type required was at hand.

The concluding talk, by Mr. C. B. SHEPPARD, was concerned with the transfer of information between a slow-speed external memory, such as magnetic wire, and a high-speed internal memory, such as acoustic delay lines or electrostatic storage tubes. Types of computer memory were tabulated and compared from the standpoint of erasibility, speed, compactness, and cost. The electrical circuitry for the transfer of signals from acoustic delay lines to magnetic tape was outlined in block form and described.

Prof. S. H. CALDWELL made a thought-provoking talk on the subject "Publication,

classification and patents." He referred to the concern of workers in the field of development of large-scale digital calculating machinery over their lack of current information concerning what their fellow workers in the field are doing. The need for free exchange of information between the groups endeavoring to develop large-scale high-speed computers was stressed. The effect upon dissemination of information of the desire of industrial interests to secure patent protection and the classification of work by military agencies was discussed. Quoting Dr. Caldwell, "It is easy to observe these influences and to call them reactionary and obstructive. They are so conspicuous that it frequently becomes easy to blame them for whatever troubles we may have. When a patent policy is too rigid and when military classification becomes unrealistic, they deserve all the blame we can muster; but I think that if we look further, we will find other major sources of difficulty and other cures than mere condemnation."

Prof. Caldwell proposed as a cure for the communication defects professional organization and a publication medium. He suggested that the National Research Council could be of assistance through two committees under the Division of Physical Sciences: the Committee on Mathematical Tables and Other Aids to Computation, Prof. R. C. ARCHIBALD, Chairman; and the Committee on High-Speed Calculating Machines, Prof. JOHN VON NEUMANN, Chairman. One result of a joint conference of these committees was the establishment in *MTAC* of the new department on "Automatic Computing Machinery."

Included in the publication of the proceedings of the symposium is a paper in absentia by Dr. LOUIS COUFFIGNAL, Centre National de la Recherche Scientifique, Paris, France. The paper concerns the extent of the application of large-scale calculating machinery. Presented diagrammatically are charts exhibiting the author's concept of a hierarchy of technical and mathematical steps leading one from a concrete problem to its final solution by fabrication or construction. A similar schematic is presented which is based on the author's classification of calculating machines and of mathematics. The central theme of the article is the effect of high-speed calculating machinery upon the systematic approach to problems requiring numerical solution.

Dr. A. T. WATERMAN began his discussion, "New vistas in mathematics," with a description of the role of the Office of Naval Research in the program of development of high-speed computation—with particular reference to automatically-sequenced, high-speed digital computers. In the early planning stages of this program, ONR suggested to the NBS that a computing center be established to serve the combined interests of industry and government agencies. Dr. Waterman mentioned plans of the NBS to establish such computing facilities on the East and West Coasts, preferably in connection with a university. The speaker emphasized the importance of a training program in the new computer techniques.

The expediency demanded by World War II precluded an extensive program of scientific research under the Office of Scientific Research and Development; however, it is hoped that, in the future, the establishment of a National Science Foundation will encourage and support research. At present support to science comes largely from the Armed Services. Because of the wisdom and judgment of these groups in administering this support, Dr. Waterman believes there is no danger of military control of science.

Emphasis was placed on the important relationship between pure and applied mathematics—specifically as applied to the formulation of a wise program of support of mathematics. To quote Dr. Waterman: "History has provided us with repeated examples of mathematical disciplines, which studied only for their intrinsic interest and dealing apparently with purely formal truths, have reached results of profound importance for our description of the physical universe." It is important to remember that the high-speed computing machine is the servant of human endeavor; the Harvard group under Prof. Aiken has kept this in mind in the solution of problems where sheer magnitude of work would have prevented solution by other devices. Heretofore scientists were able only to prove the possibility of solution of certain problems; now, with the advantageous speeds promised by these new computers, there can be greater emphasis on actually solving the problem and in turn opening up new vistas on unsolved questions and encouraging development of new and penetrating theories.

The review of the contents of the talks presented at the Harvard Symposium, even though only a very brief abstract of each talk has been given, has been an extended one. It is believed that this fact should occasion no disquietude of reviewers and editors since in this case the length of the review can be interpreted as a measure of the importance of the material reviewed. The symposium was very well organized and the caliber of the participants beyond reproach. The high stature of the speakers is evidenced by the agreement between their prognostications concerning future developments in the field of high-speed calculating machinery and the direction this development has followed since the symposium. In view of the fact that the ENIAC remains the only operating large-scale electronic digital computing machine, one might tend to deplore mildly the optimism of certain speakers concerning the time required for the consummation of the developments under way in the field. However, no one would insist that their "batting average" on estimating the duration of engineering-development programs was an unusually low one.

This volume is recommended reading for everyone interested in the development and application of large-scale computing machinery.

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[1] For details, see F. L. ALT, "A Bell Telephone Laboratories' Computing Machine," *MTAC*, v. 3, p. 1–13, 69–84.

[2] This machine has since been completed and is now in successful operation at the Naval Proving Ground, Dahlgren, Va.

[3] In our program as published we had here the name of Dr. K. G. MACLEISH.

5. INSTITUTE FOR ADVANCED STUDY, Princeton, N. J. *Second Interim Progress Report on the Physical Realization of an Electronic Computing Instrument*, by JULIAN H. BIGELOW, THEODORE W. HILDEBRANDT, JAMES H. POMERENE, RICHARD L. SNYDER, RALPH J. SLUTZ & WILLIS H. WARE. 1 July 1947, 48 leaves, 52 figs., 12 drawings, 3 tables. 21.6 × 27.9 cm.

This report covers conditions sufficiently far in the past that it is of little value in indicating the present status of computer development at the Institute for Advanced Study. Approximately half of the report concerns details of magnetic wire drive and performance. The bulk of the remainder of the report covers experimental results on various circuit components planned for the IAS Computer, such as "flip-flops," "registers," "accumulators," and "pulse drivers."

Since the report deals almost exclusively with the special features pertinent to the particular computer visualized at the Institute for Advanced Study, it will be of interest chiefly to those connected with this project or to those who are interested in the design and construction of similar components. A large portion of the detail on magnetic wire performance, for example, will not be of great value to one who is primarily interested in the use of magnetic tape rather than wire. The report is probably of more general interest in its treatment of the methods of approach to the problems encountered. Some of these methods will doubtless suggest new viewpoints to the reader faced with analogous problems.

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6. JOHN B. IRWIN, "The expected performance of the EDVAC on some astronomical problems," Astron. Soc. Pacific, *Publs.*, v. 60, 1948, p. 235–244. 15.2 × 22.8 cm.

The EDVAC (Electronic Discrete Variable Computer) is a comparatively small, extremely fast electronic digital computer now being tested at the Moore School of Electrical Engineering, University of Pennsylvania. Because of its great speed, it is valuable in solving astronomical problems heretofore considered too difficult or laborious to attempt. This article briefly describes the essential design features of the machine and its application to some of the above-mentioned problems. In most of the examples discussed here, over one-half of

the estimated time of solution is spent in input-output time, and therefore these problems could be handled by slower computers, leaving the EDVAC free to tackle the more time-consuming problems. The successful application of these machines to astronomical problems, the report concludes, will depend on the availability of these machines to astronomers, the quality of mathematical personnel, and the efficiency of programing.

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7. G. A. Korn, "Elements of d-c analog computers," *Electronics*, v. 21, 1948, p. 122–127, bibl. 20.3 × 27.9 cm.

"Design criteria of simple circuits for adding, multiplying, integrating and differentiating are presented with their limits of accuracy. Operating principles and types of applications of direct-current electrical analog computers are summarized."

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8. K. G. MacLeish, R. D. O'Neal, & A. W. Tyler, *High-Speed Digital Electronic Computer*. Eastman Kodak Co., Rochester, N. Y. 22 Feb. 1946, 24 leaves. 21.6 × 27.9 cm.

Early in 1946, the Eastman Kodak Company issued a proposal for designing and constructing a computer that would embody several significant advances over the designs proposed up to that time. Among these are: 1) the use of photographic film as an input-output medium, capable of storing from one hundred to five hundred 30-binary-digit words per inch, 2) methods for scanning a two to three foot loop of film at rates up to a million words per second, 3) a multiplying unit capable of obtaining the product of two 30-binary-digit numbers in 50 μsec. (microseconds), 4) graphical output on a cathode-ray tube screen, 5) automatic execution of floating binary-point operations.

The report first lists the fundamental desiderata for a computing machine, namely: reliability, speed, ease of computation, and flexibility. It then describes ways in which each of these may be achieved. Reasons are given why the Eastman Kodak Company prefers the binary to the decimal number system, parallel to sequential operation, and photographic film to magnetic tape as an input-output medium.

A suitable memory device, according to the report, would have an access time of only 10 μsec. per word, in which the word would not be erased when transferred. Such storage must be achieved, however, without an unreasonable amount of gadgetry. No definite decision had been made, at the time of writing, on the type of memory device; but a detailed discussion covers the pros and cons of storing numbers by: (1) a charge in capacitors in electron tubes; (2) current in gas tubes; (3) position in special beam tubes; (4) circulating pulses in delay lines; and (5) stable states in trigger pairs.

The proposed machine would handle numbers in the normal form, $N = q2^p$, where q consists of 30 binary digits and the exponent p has a maximum of six binary digits. 36 trunk lines would transfer such numbers within the computer. The control is based on a three-address system so that a program word would contain, besides the operation, a Source (address of word stored in the High-Speed Memory), a Destination (address of the Computing Unit), and a Next Command (address of word stored in the High-Speed Memory). In addition to the High-Speed Memory, the principal parts of the computer would be

a) a Film Preparation Unit, containing provisions for introducing decimal data (representing numbers or orders), either from a key board or punched-card reader, for translating information into binary codes and for photographing these data on 35-mm film;

b) Film Readers, capable of introducing information into the High-Speed Memory at five hundred to one thousand words per second;

c) Computing Units, capable of carrying out floating-point addition and subtraction in about 20 μsec, multiplication in 50–250 μsec, division and extraction of square root in 100–250 μsec;

d) a Cyclic Program Unit, containing a high-speed film reader capable of scanning a

short loop of tape at speeds of up to a million words per second, this loop carrying either a cyclic series of program data or a function table with appropriate coefficients for interpolation, so that a "look-up" may be achieved in from .1 to 10 milliseconds;

e) the Control, receiving program data either from the Cyclic Program Unit or from the High-Speed Memory, and interpreting this information for the computing units;

f) Output Units, containing a graphical device yielding about 1 percent accuracy; a binary-to-decimal translator, as well as decimal printer yielding up to nine significant figure accuracy at the rate of some 15 numbers per second; and a recorder of binary output on photographic film at the rate of about 1000 words per second.

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9. MOORE SCHOOL OF ELECTRICAL ENGINEERING, Univ. of Pennsylvania, *Theory and Techniques for Design of Electronic Digital Computers.* Lectures delivered 8 July–31 August 1946. V. 3–4, mimeographed, Philadelphia, 30 June 1948. **A.**, v. 3, 157 leaves, Lectures 22, 23–24 (abstracts), 25, 27–29, 31, 33; **B.** v. 4, 158 leaves, Lectures 34, 35, 37, 39, 43–45, 46 (abstracts) 47. 21.5 × 28 cm., $10. V. 1–2 were reviewed *MTAC*, v. 3, p. 128–132.

A. At the time the lectures were delivered there was little general knowledge concerning automatic digital computing machinery. Unfortunately, however, in a field progressing as rapidly as is that of electronic computers today, a delay of two years in publication substantially vitiates the usefulness of a report. This delay coupled with the incompleteness of several of the lectures has now made this report primarily of historical interest.

Lecture 22. *Sorting and collating* by J. W. MAUCHLY. After showing that comparison operations are essential to an automatic machine design for handling large computational problems, the author shows how these facilities may be used to change an initially random sequence of variables into an ordered one. Two cases are considered: (1) that in which the desired ordered sequence is monotonic throughout; and (2) that in which the variables are to be ordered only by classes.

On modern electronic machines the operating speed is much higher for operations involving only the internal high-speed memory than it is for operations involving the transfer of data to or from an external memory, such as magnetic storage. Under these conditions the time of sorting for a number of data much larger than the internal memory can accommodate becomes largely the time necessary for external memory transfers. On this basis the author analyzes the comparative times of different procedures. Considerable attention is paid to performing decimal sorting on a machine having a minimum of input and output tapes.

Lecture 25. *Conversion between binary and decimal number systems* by J. W. MAUCHLY. In many of the standard algebraic treatments of the conversion from one number system to another, no attention is paid to carrying out all of the operations in only one of the two number systems. In the design of an automatic computer to work in any system other than the decimal, it would be highly inefficient to build arithmetic circuits working in both systems. The author shows how the arithmetic operations can be carried out completely within the machine's number system, both for conversion from decimal to binary and for conversion from binary to decimal. He also points out that, provided the numbers entered into the machine are either wholly integral or wholly fractional, one of these conversions will involve only a particularly simple multiplication while the other will involve division. He goes on to show that by including an extra multiplication in one of the conversions the remaining operations become entirely simplified multiplication. This analysis is of particular interest if it is desired either to perform a conversion in minimum time with the regular arithmetic facilities of the machine, or if it is desired to build special conversion equipment of minimum complexity.

Lecture 27. *Magnetic recording* by CHUAN CHU. Equations are derived to describe the

recording and reproducing processes on a magnetic medium. For the recording processes, the author points out that the non-linearity of the magnetic medium invalidates the linear analyses customarily applied in analyzing acoustic recordings. He then continues, however, to derive equations for the recording processes which would be useful for analyzing pulse recording only if a linear substitution were to be applied. For reproduction, however, the magnetic fluxes generated in the reproducing head are in general so low that a linear analysis is well warranted. For this case the author makes an excellent analysis of the effect of gap width in degrading the reproduced signal.

Lecture 28. *Tapetypers and printing mechanisms* by J. P. ECKERT, JR. This lecture gives a discussion of the author's preferences concerning typing and printing mechanisms, with a description of possible methods of using gang printers to obtain high-speed output from a machine.

Lecture 29. *A review of government requirements and activities in the field of automatic digital computing machinery* by J. H. CURTISS. This lecture gives an excellent and apparently comprehensive survey reviewing the history of automatic digital machines, describing those which were constructed up to the time of the lecture, discussing those which were then planned, and giving considerable attention to the need for such equipment in governmental operations.

Lecture 31. *Numerical mathematical methods—VIII* by ARTHUR W. BURKS. This lecture reviews the method of "closed-cycle" integration of total differential equations in which at each stage only the values of the variables for the next preceding stage are used, with no reference to earlier stages. Approximation formulae of the first through the fourth order are discussed, covering the Heun method and the Runge-Kutta method. The author points out that in the Runge-Kutta method one is actually generating intermediate values of the variable, making this method appear at first to be no improvement in machine capacity over open-cycle integration of the same order. He shows, however, that the work can be so arranged that the intermediate values can be computed and used one by one, without the need for storing them simultaneously in the machine's memory.

Lecture 33. *Continuous variable input and output devices* by J. P. ECKERT, JR. The author states that there are four fields of primary utility for the application of digital computing in conjunction with continuous variable input and output devices. These are: gun directors, guided missiles, industrial control, and real time simulators ("trainers"). In addition to a general discussion of the utility of these operations he describes some rather interesting ideas for carrying out the conversion. One particular idea is a means for controlling a cathode-ray tube so that the position of the spot depends upon the previous history of the signals applied to the equipment. This is accomplished by enclosing appropriately perforated plates in the cathode-ray tube and using a feedback system to give the plates partial control of the spot position. If such a scheme were proved to be feasible it could be used to provide discrete indications within the cathode-ray tube for many possible values other than the two-condition storage carried out in the customary flip-flop, and in addition other variations of this scheme could be used to carry out addition or other computations compactly and at high rates of speed. A slightly different application of the method would provide an absolute positioning scheme for locating the spot in the tube; this would be useful in some types of memory tube applications. At the present time this lecture probably holds the most interest of any in the volume, since through the pressure of other work little has been done in the field described here. It holds promise for significant developments in the future.

RALPH J. SLUTZ

NBS

B. Lecture 34. *Reliability and checking in digital computing systems* by S. B. WILLIAMS. This is a discussion of reliability in checking mainly applicable to the Bell Relay Machine. This machine has a very complete low-level checking system based on its property of having only one relay closing in a given set. It also has the feature of step-by-step operation wherein each sequence operation must be checked before the next operation can proceed. The reading of the paper tape is checked by a redundant coding, now common practice. The chief con-

clusion is that a machine failure should give an alarm and hold the results for diagnosis and resumption of calculation.

Lecture 35. *Reliability and checking* by J. P. Eckert, Jr. This gives only general considerations on checking, such as the truism that no amount of checking will increase machine reliability, and that checking is a second line of defense. It is pointed out that reproducible errors are easy to find by means of test runs. Intermittent errors cause more difficulty, but a fundamental advantage of serial machines is that all digits are affected so that intermittent errors are apt to be easily detected in a smoothness test. This test is characteristic of so-called high-level checking as opposed to the low-level checks of the relay machine.

Lecture 37. *Code and control—II. Machine design and instruction codes* by J. W. Mauchly. Here is a discussion of the concept of a general purpose machine and the problem of the optimum number of orders. Too few orders make for difficult and lengthy programs, while, on the other hand, too many orders make the programing difficult, since too many things must be remembered and the machine then becomes very complex and therefore less reliable. General types of orders are discussed, and some comments are offered on the relative length of other words and number words from the standpoint of efficient storage.

Lecture 39. *Code and control—IV. Examples of a three-address code and the use of 'stop order tags'* by Calvin N. Mooers. There is a discussion of Tags for terminating subsequences, as, for example, when a nonanalytic boundary of a region of integration is reached. Two problems are coded in full for illustration.

Lecture 43. *The Selectron* by Jan Rajchman. This presents physical principles and problems of the Selectron memory tube together with an analysis of the combinational system of element selection used.

Lecture 44. *Discussion of ideas for the Naval Ordnance Laboratory computing machine* by C. N. Mooers. This is a discussion of some design proposals of an EDVAC-type machine.

Lecture 45. *A parallel-channel computing machine* by J. P. Eckert, Jr. This discusses the speed advantages and calculation disadvantages of a parallel-channel type computer as compared with single-channel serial operation.

Lecture 46. *A four-channel, coded-decimal electrostatic machine* by C. B. Sheppard. There is a summary of a lecture on some design considerations for a machine using four memory tubes containing 12,000 storage elements each.

Lecture 47. *Description of serial acoustic binary EDVAC* by T. K. Sharpless. Block diagrams of the EDVAC are presented and its operations illustrated by coding the iterative method of finding a reciprocal and tracing the operations through the machine.

Chester H. Page

NBS

10. R. D. O'Neal & A. W. Tyler, *Progress Report no. 1, Photographic Digital Reader-Recorder*. Eastman Kodak Co., Rochester, N. Y. 7 June 1948, 23 leaves. (Contract N6ori-205 with Office of Naval Research, Special Devices Center). 21.6 × 27.9 cm.

This report discusses the objectives and techniques in designing and constructing a photographic input-output device to insert and receive information on the M.I.T. Servomechanism Laboratory's PROJECT WHIRLWIND Computer. Among the input-output requirements discussed are

1) the necessity that the film must be handled at a variety of speeds; 2) the storage of both a number and its complement on film in binary form, a 1 being represented by a clear spot and a 0 by an opaque spot; 3) the necessity that the machine be able to move tape in either direction to search efficiently for information.

The same piece of equipment will read and record. Recording, or exposure of the film, is accomplished by a masked cathode-ray tube, while reading is done with a phototube and light source. A detailed description of these operations is illustrated with block diagrams. Four possible arrangements of data on film are presented together with a method of recording without complements. "Breadboard" experimental results and problems are discussed, as

well as results of tests on various components, such as magnetic clutches, phototubes, cathode-ray tubes, and r-f power supplies and suitable film drives.

The summary includes the following information: 1) Reading and recording will take place at a peak rate of 2000 25-binary-digit words and complements per second. 2) One hundred 25-binary-digit words and complements will be stored on one inch of 35-mm unperforated film stock. 3) All information recorded or read will be automatically checked against the original. 4) A minimum of optical and electronic adjustments will be necessary, and all panels will be readily removable for servicing. 5) Since film will be in daylight-loading magazines, short lengths may be removed without exposing film in the main part of the drive. 6) Commercially available automatic film processing machines will be used with the above equipment. The film must be removed from the reader-recorder and placed in the processor for development.

MDL

11. F. C. Williams & T. Kilburn, "Electronic digital computers," *Nature*, v. 162, 25 Sept. 1948, p. 487. 17.8 × 25.4 cm.

Presented here is a very brief description of a small electronic digital computing machine built to test the soundness of a storage principle. The computer is now in successful operation at the Royal Society Computing Laboratory, Electrical Engineering Laboratories, University, Manchester 13.

MDL

12. U. S. Air Force, Planning Research Div., *Scientific Planning Techniques, Project SCOOP* [Scientific Computation of Optimum Programs]. *Discussion Paper*, no. 1-DU, 5 Aug. 1948, 29 p., 9 charts. 20.3 × 26.7 cm.

The scope, policies, and administration of Project SCOOP are set forth in Air Force, *Letter* 170–3, dated 13 Oct. 1948. The following is a quotation from this *Letter:* "The primary objective of Project SCOOP is the development of an advanced design for an integrated and comprehensive system for the planning and control of all Air Force activities. The recent development of high-speed digital electronic computers presages an extensive application of mathematics to large-scale management problems of the quantitative type. Project SCOOP is designed to prepare the Air Force to take maximum advantage of these developments. The basic principle of SCOOP is the simulation of Air Force operations by large sets of simultaneous equations."

MDL

News

Association for Computing Machinery.—The ballot for election of President, Vice-President, Section Officers, and Members-at-large for the period ending May 31, 1949, under the provisional Constitution and Bylaws, has been submitted to the membership with the Secretary's October 21st report. Because the nominee for the Section Officer from New York, Dr. Samuel Lubkin, has moved to Washington, and is now with the NBSAML, the nominating committee has made a substitute nomination: Mr. E. G. Andrews (BTL, now on the ACM Council).

It was suggested that the Association issue a more regular bulletin than the present irregular series of reports. The Council has invited the views of the members on this question and volunteers for the work it will involve, but the Secretary has received no comments on this suggestion. This would seem to indicate that there is no substantial demand for such a bulletin and that possibly *MTOAC* largely fills this need for the Association.

A copy of a mimeographed summary of any one of the reports will be sent to any member who has not already received a copy, upon written request to the Secretary, Mr. Edmund C. Berkeley, 36 West 11 Street, New York. The summaries prepared are as follows: The Pilot Model of EDVAC, by T. K. Sharpless, 2 pp., 9/22/47. Optimum Size of Automatic

Computers, by G. R. STIBITZ, 2 pp., 12/24/47. Operating Characteristics of the Aberdeen Machines, by F. L. ALT, 2 pp., 12/29/47. Reduction of Doppler Observations, by DORRIT HOFFLEIT, 1 p., 1/11/48 (see MTAC, v. 3, p. 373–377) General Principles of Coding with Applications to the ENIAC, by J. VON NEUMANN, 1 p., 1/17/48. Adaptation of the ENIAC to von Neumann's Coding Technique, by R. F. CLIPPINGER, 2 pp., 3/15/48. Census Applications for High-Speed Computing Machines, by J. L. MCPHERSON, 1 p., 3/30/48 (see MTAC, v. 3, p. 121–126). The Raytheon Computer, by R. V. D. Campbell, 2 pp., 4/6/48.

Naval Research Lab., Washington D. C.—On 17 Nov. 1948, Prof. H. H. Aiken, of the Harvard University Computation Laboratory, discussed design features and operational characteristics of the Mark I, II, and III relay computers. The discussion was supplemented with slides.

The first two computers were developed at Harvard University under the leadership of Prof. Aiken and have been placed in service—the Mark I at Harvard and the Mark II at the Naval Proving Ground, Dahlgren, Va. Up to the present time they have performed with excellent reliability (the Mark I has averaged 60 to 75% successful operation and in some cases as high as 95%, and the Mark II has averaged about 85%). A comparison of the operation speeds of the two machines was given as follows:

	Mark I (23 dig. nos.)	Mark II (10 dig. nos.)
Multiplication	5 seconds	750 milliseconds
Addition	300 milliseconds	200 milliseconds

In addition it was pointed out that the Mark II machine could be mathematically cut in half (a valuable facility in the case where trajectories, arising so frequently in the work at Dahlgren, are to be handled).

The talk was highlighted by a discussion of the Mark III calculator, now being developed by Professor Aiken's group. It is expected that this machine will also be available for operation at Dahlgren, by June 1949. The fundamental components of the machine, like those of the Mark II, are the latch, the 2-coil, and the 3-coil relays. It will use a magnetic drum memory and will have a memory capacity of approximately 4000 16-digit-numbers. Magnetic tape is to be used in the eight input-output tape mechanisms. The addition and multiplication times quoted for the Mark III were 4 milliseconds and 12.5 milliseconds, respectively.

Although the machine operation speeds mentioned are comparatively small, Prof. Aiken believes that they are sufficiently fast until more is learned about numerical methods to be used in problem programing. In the opinion of the speaker, one should strive for more expedient problem preparation techniques rather than for increased speed in the machine.

Office of Naval Res., Washington, D. C.—On 15 Dec. 1948, an interesting and informative lecture (with slides) on the Mark II Calculator was given by Dr. C. C. BRAMBLE, director of Computation and Ballistics, NPG, Dahlgren, Va. The speech paralleled that of Prof. AIKEN, although Dr. Bramble presented a more detailed description of many of the machine design features and a step-by-step explanation of a particular coding routine which had been used on the machine.

Errata.—*MTAC*, v. 3, p. 216, l. 8, *for* both of King's College, *read* respectively of Birbeck College Res. Labs., and The British Rubber Producers' Res. Labs.

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z–VII

1. J. R. BOTHEL, "Slide Rule easily made for converting ram to sample pressures," *Chem. Engin.*, v. 55, Sept. 1948, p. 125–126. 21 × 28.5 cm.

In the use of hydraulic presses for different materials or under various pressure conditions, it is necessary to convert the pressure on the rams to pressure on the sample.

2. EDWIN A. GOLDBERG, "Details of the simultaneous equation solver," *RCA Review*, v. 9, Sept. 1948, p. 394–405, 14.6 × 22.5 cm.

"An electronic device for solving systems of linear simultaneous equations such as those encountered in circuit analysis work, quantitative chemical analysis, and a wide range of physical problems is described in this paper. Emphasis is placed on the actual electrical design employed in the execution of a practical model and the operation of the device is considered." See also *MTAC*, v. 3, p. 329–330, Bibl. Z-V, 2.

3. ARTHUR C. HARDY & EDWARD C. DENCH, "An electronic method for solving simultaneous equations," Optical Soc. Amer., *Jn.*, v. 38, 1948, p. 308–312. 19.6 × 26.6 cm. See *Math. Revs.*, v. 9, 1948, p. 535.

4. LO-HO, "Construction of alignment nomogram from empirical data," Franklin Inst., *Jn.*, v. 245, 1948, p. 227–244. 16 × 24.2 cm. See *Math. Revs.*, v. 10, p. 621 (R. CHURCH).

5. CARL P. NACHOD, "Nomograph for the square root of the sum of squares," *Product Engin.*, v. 19, Nov. 1948, p. 155. 21 × 28.4 cm.

6. F. K. RUBBERT, "Zur Radizierung mit der Rechenmaschine," *Z. angew. Math.*, v. 28, June 1948, p. 190–191. 20.8 × 29.4 cm.

7. RUFUS F. STROHM & ARCHIBALD DEGROOT, *The Slide Rule. How to Use It.* Second ed. Scranton, Pa., International Textbook Co., 1948, viii, 95 p. 13 × 21 cm. First ed., 1939.

Extract from Preface: "No attempt has been made to show all the various forms of slide rules or to explain all the ways in which they may be used. However, the fundamental principles underlying the operation of the types of slide rules that are in common use are explained fully, and a person who knows how to operate these types will have no difficulty in learning how to use a slide rule intended for a special purpose."

"Slide rules with folded scales and log log scales," p. 68–84.

8. ANTONÍN SVOBODA, *Computing Mechanisms and Linkages, edited by* HUBERT M. JAMES. Office of Scientific Research and Development, National Defense Research Committee. (M.I.T. *Radiation Lab. Series*, no. 27), New York, McGraw-Hill, 1948, xii, 359 p. + plate in pocket. 15.1 × 22.7 cm. $4.50.

This book deals specifically with the analytical design of bar-linkage elements for continuously acting computing mechanisms. Bar linkages have many advantages in this application, particularly with respect to compactness and cost. They are limited somewhat in the field of functions covered and in inherent structural error. If an inherent error of not over 0.3% is tolerable, it is relatively easy to design a linkage computer; to reduce the error below 0.1% is relatively difficult. The author presents extensive numerical tables and an intersection nomogram (in the pocket) to aid in the design of practical slider-crank and "three-bar" linkages, which singly or in combination may be used to generate the functional relation between two variables within given tolerances. The author also presents a design procedure for developing star linkages to generate functions of three variables. The detailed mathematical design is usually difficult and laborious. Practical mechanisms rarely fit exactly the function to be mechanized; in order to obtain the desired degree of fit between generated and given functions, it becomes necessary to adjust a number of linkage parameters by a method of successive approximations. The author has rendered a distinct service

in setting up rational design procedures and in removing much of the drudgery involved in developing preliminary approximations. He has obviously taken great pains to present clearly and concisely a subject in which confusion and misdirected effort might easily attend the inexperienced designer. The book is exceptionally well written; the material is logically and harmoniously developed. Designs are carried out for each type of mechanization considered.

There are ten chapters and two appendices in the book. The intersection nomogram for designing three-bar linkages is inserted in the back cover. In Appendix B (p. 333–352) the author has tables from which an enlarged nomogram may be constructed for design purposes.

A brief review of standard continuous computing elements in Chapter 1 is followed in Chapter 2 with a brief survey of bar-linkage computing elements, including harmonic transformers (slider-crank mechanisms), the more versatile three-bar linkage, bar-linkage adders, bar-linkage multipliers and dividers, and combinations of these elements for solving any problem that can be expressed in a system of equations involving only these operations. Chapter 3 defines the terminology to be used for linkage parameters and for the related variables.

Chapter 4 discusses in considerable detail the use of harmonic transformers for mechanizing over a limited range functions which are sinusoidal or approximately sinusoidal. The ideal harmonic transformer (infinitely long connecting link) serves as the starting point for the design of a nonideal (finite) harmonic transformer. Use is made of the tables in Appendix A (p. 301–332) for a rapid determination of the ideal transformer and for a rapid evaluation of the structural error involved in going to the nonideal transformer.

Chapter 5 discusses the design of three-bar linkage computing elements. It illustrates thirty-two curve classifications generated by such linkages. The relations between crank positions and the geometrical properties of the linkage are incorporated in an intersection nomogram. This serves as a powerful tool in designing linkages to mechanize given functions when it is possible to preassign values for two of the design constants. The author also develops a geometric method for designing linkages in the rare case that only one design constant can be preassigned.

Only rarely can one mechanize a given function with high accuracy by means of individual elements. Chapter 6 discusses combinations of elements. The nonideal double transformer has seven adjustable parameters as against five for the three-bar linkage and four for the nonideal single transformer. If a three-bar linkage is interposed between two transformers, the number of adjustable parameters becomes twelve. The double three-bar linkage is governed by nine parameters. Such combinations obviously can mechanize satisfactorily a much greater field of functions.

Chapter 7 discusses the final adjustment of linkage constants. The accuracy of the graphical methods hitherto discussed is limited; when high accuracy is required the final adjustment must be carried out by numerical methods. The generated function is made to fit a predetermined set of precision points analytically. Adding eccentric linkages serves to increase the number of adjustable parameters and hence precision points.

A function of three variables may be represented by a grid structure consisting of three families of curves such that a curve of each family passes through every point of intersection. Chapters 8, 9 and 10 discuss the mechanization of such a grid structure. The basic idea is to make use of a topological transformation which transforms the given grid structure into a form which suggests a satisfactory mechanical form of grid generator. The star linkage is a satisfactory grid generator.

Any functional relation that can be generated by a star linkage can also be represented by an intersection nomogram consisting of three families of circles. Consequently the desired transformation should carry the given grid structure into this form, or into one closely approximating it in a limited region. Final adjustment of the constants is carried out analytically as in the other linkages.

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NOTES

99. DANIEL FRIEDRICH ERNST MEISSEL.—There have been several Germans who, while teaching in secondary schools, contributed something notable to mathematics. KARL WEIERSTRASS (1815–1897) taught in secondary schools until he was 41 years old, when, after the publication of important papers in Crelle's *Jn.*, he was appointed a professor of mathematics at the University of Berlin. HERMANN G. GRASSMANN (1809–1877), who spent his life as teacher in a Stettin Gymnasium, was a man of extraordinary versatility, and author of *Ausdehnungslehre* (1844, 1862), a contribution towards the vector analysis of a later generation. Dr. JOHANNES TROPFKE (1866–1939), long director of the Kirschner Oberrealschule in Berlin, wrote the best reference work in existence for the history of elementary mathematics (1902–1940; there was a second ed. of the 7 v. and a third ed. of v. 1–4). We shall now assemble some of the facts concerning the life and publications of another man whose main life work was in secondary schools.

MEISSEL was born in Neustadt-Eberswalde 31 July 1826, and died at Kiel 11 March 1895. His education was begun in the Friedrich Wilhelm-Gymnasium in Berlin and continued as he studied mathematics under JACOBI at the University of Berlin (1847–1850) and got his doctorate at the Univ. of Halle in 1850 (Diss.: *De serie quadam Jacobiana*). During 1852–56 he was in Berlin a docent at the Bergakademie and Bauakademie. From 1856 to 1871 he was director of the Provinzial-Gewerbeschule at Iserlohn, and from 1871 to the time of his death director of the Ober-Realschule in Kiel.

He was the author of two text-books, one on the calculus (1854), and the other on arithmetic and algebra (1861), and of various articles and tables published in Crelle's *Jn.*, Grunert's *Archiv*, Poggendorff's *Annalen*, *Astr. Nachrichten*, *Math. Annalen*, Progr. Iserlohn (three, 1862–70), and Progr. Kiel (twelve, 1874–1894). These Programmen are listed in E. WÖLFFING, *Mathematischer Bücherschatz* (1903), and other publications in the Royal Soc. *Catalogue of Scientific Papers*, and in "POGGENDORFF," v. 2–4. We now present a chronological list of Meissel's mathematical tables.

1_1. *Sammlung mathematischer Tafeln berechnet und herausgegeben. Erste Lieferung* [:*Tafel der Elliptischen Functionen enthaltend die Werte von Log. Vulg. q auf acht Decimalen für das von Minute zu Minute Fortschreitende Argument*]. Iserlohn, Selbstverlag, 1860, ii, 20 p. No more published. $\log q$, θ = [0(1′)90°; 8D]. "The largest single-entry table of elliptic functions in existence"; see *MTAC*, v. 3, p. 275–276; for this, and errors.

1_2. J. L. F. BERTRAND, *Traité de Calcul Différentiel et de Calcul Intégral*, Deuxième partie, *Calcul Intégral*. Paris, 1870, p. 711–717. FLETCHER shows (*MTAC*, v. 3, p. 261) that (because of its errors) the 5D table of $\log q$, θ = 0(5′)90° was an unacknowledged abridgment of Meissel's table. He also points out that this table of Bertrand was then copied by LÉVY (1898), and POTIN (1925).

2_1. *Tafel der Bessel'schen Functionen* I_k^0 *und* I_k^1 *von* $k = 0$ *bis* $k = 15.5$ *berechnet*. Akad. d. Wissen., Berlin, *Abh.* for 1888, Berlin, 1889, p. 4–23. A table of $J_0(k)$ and $J_1(k)$, k = [0(.01)15.5; 12D]. [For errors in $J_0(.62)$, $J_0(1.71)$, $J_0(1.89)$, $J_1(7.87)$ see *MTAC*, v. 1, p. 298.] The first 10 zeros of $J_0(x)$, to 10D, are given on p. 3.

2_2. A. GRAY & G. B. MATHEWS, *A Treatise on Bessel Functions and their Applications to Physics*. London, 1895 [GRAY & MATHEWS, 1895], p. 247–266, 244; second ed. prepared by A. GRAY & T. M. MACROBERT, London, 1922; reprinted 1931 and 1936 [GRAY, MATHEWS & MACROBERT, 1922], p. 267–286, 300. For errors in both eds. in $J_0(.62)$, $J_0(3.07)$, $J_1(7.87)$, and in the first edition $J_0(1.89)$, $J_0(5.90)$, see *MTAC*, v. 1, p. 290, 298.

2_3. E. Jahnke & F. Emde, *Funktionentafeln*, 1909; 4D abridgments, p. 111–123; reprints 1923, 1928. Second ed., 1933, p. 228–235, 237. Third ed., 1938, and 1945, p. 156–163, 166.

2_4. The zeros of $J_0(x)$ are included in **3_3**.

3_1. *Über die Bessel'schen Functionen I_k^0 und I_k^1*. Ober-Realschule in Kiel, *Jahres-Bericht 1889–90*, Kiel, 1890, p. 4. Tables of the first 50 zeros of $J_1(x) = 0$, with corresponding values of $J_0(x_n)$, each to 16D.

3_2. Gray & Mathews, 1895, p. 280; Gray, Mathews & MacRobert, 1922, p. 301. Meissel's correct value for $x_1^{(9)}$ was copied incorrectly in both editions.

3_3. H. T. Davis & W. J. Kirkham, "A new table of the zeros of the Bessel functions $J_0(x)$ and $J_1(x)$ with corresponding values of $J_1(x)$ and $J_0(x)$," Amer. Math. Soc., *Bull.*, v. 33, 1927, p. 769–770. A 10D rounding off of Meissel, and other material. There is a 9-unit error in the tenth decimal of $x_1^{(9)}$.

3_4. The Meissel values of **3_3**, corrected, were reprinted in BAASMTC, *Bessel Functions*, Part 1, 1937, p. 171.

3_5. E. Jahnke & F. Emde, *Funktionentafeln*, 1909; 4D abridgments of $x_0^{(s)}$ and $x_1^{(s)}$, and 4S of $J_0(x_1^{(s)})$, p. 122–123; reprints, 1923, 1928. Second ed., 1933, p. 237. Third ed., 1938, and 1945, p. 166.

4. "Abgekürzte Tafel der Bessel'schen Functionen $I_k^{(h)}$ (Auszug aus einer grösseren Tafel mit 18 Decimalen)," *Astr. Nach.*, v. 128, 1891, cols. 153–156. Mainly a 6D abridgment of no. **7**, of $J_k(h)$, $h = 1(1)10, 16, 20$; $7 \leq k \leq 35$. Also 8D values of $J_k(1000)$ for $k = 967, 968, 981(1)1000$. For errors when $k = 967$ and 968, see *MTAC*, v. 2, p. 47–48.

5. "Neue Entwickelungen über die Bessel'schen Functionen," *Astr. Nachrichten*, v. 129, 1892, cols. 283–284. Table of $10^8 \cdot J_{2n}(n)$, $n = [10(1)14; 8D]$, $[15(1)19; 10D]$, $[20, 21; 12D]$.

6. *Entwurf einer Tafel aus welcher die sechs Elemente einer beliebigen Menge sphärischer Dreiecke sofort entnommen werden können.* Ober-Realschule in Kiel, *Jahres-Bericht* 1893–94, Kiel, 1894, p. 1–7. See *Astron. Nachrichten*, v. 95, 1879, col. 69–74.

7_1. Tables of $J_n(x)$, $x = [1(1)24; 18D]$, $n = 0(1)N - 1$, $17 \leq N \leq 61$, Gray & Mathews, 1895, p. 266–279; Gray, Mathews & MacRobert, 1922, p. 286–299. These tables were first published here in 1895. There are 5 errors; the following 4 are noted in *MTAC*, v. 1, p. 290: $J_4(5)$, $J_{23}(6)$, $J_{30}(14)$, $J_{31}(16)$. In FMR, *Index*, p. 246 Dr. Miller notes that in $J_{26}(6)$ the last three digits, 415, should read 507. In recently published Harvard tables $J_n(x)$, for $n = 0(1)39$ and $x = [1(1)99; 10D]$ are given. But J. C. P. Miller & C. E. Gwyther are extending Meissel's table to cover the range $x = [0(1)100; 18D]$.

7_2. Jahnke & Emde, 1909, 1923, 1928, 4S abridgment, p. 149–157; $17 \leq N \leq 61$. Second ed., 1933, p. 242–249. Third ed. 1938 and 1945, p. 171–177. Error in $J_9(21)$ except in 1945 ed. In this edition however there are 3 other errors: $J_{23}(6)$ and $J_{31}(16)$ should have their last digit values changed by unity, and in $J_{26}(6)$ *for* $.0^{14}4415$, *read* $.0^{14}4507$.

7_3. J. W. Strutt, Baron Rayleigh, "The problem of the whispering gallery," *Phil. Mag.*, s. 6, v. 20, 1910, p. 1002; also in his *Scientific Papers*, v. 5, 1912, p. 618; 4D abridgment of $J_{18}(x)$, $J_{21}(x)$, $x = 11(1)24$.

For a copy of the portrait of Dr. Meissel (1890) which we have reproduced we are greatly indebted to Dr. W. D. C. Danielson, director of the Humboldt Schule in Kiel. The portrait is a copy of the one hanging in the Aula of the Schule. This portrait was secured through the friendly cooperation of Professor Fritz Emde.

Apart from sources, mentioned above, concerning Meissel and his work, we may note the following:

1. C. N. A. Krueger (1832–1896), "Todes-Anzeige," *Astron. Nachrichten*, v. 137, 1895, col. 239–240. Krueger here told readers of the *A.N.* that he had a large number of the Meissel's 1860 tables (no. **1**) which he could place at their disposal!

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2. *Leopoldina*, Halle, v. 31, 1895, p. 102. Brief note.
3. Ober-Realschule in Kiel, *Jahres-Bericht*, 1894–95, Kiel, 1895.
4. J. H. ECKARDT, *Aus der Schuljungenzeit. Erinnerungen an den Buchwaldschen Hof*, Kiel, 1911. Reported by Dr. DANIELSON.

R. C. A.

100. A NEW FACTORIZATION OF $2^n + 1$.—In a letter dated 20 Dec. 1948, AIMÉ FERRIER (b. 6 May 1896), Principal of Collège de Cusset, Allier, France, sent us the following communication:

"J'ai établi successivement:

(i) que $N = \frac{1}{3}(2^{67} + 1)$ est composé [9.X.48], en appliquant la reciproque de la contraire du theorème de FERMAT. LEHMER ayant établi qu'aucun nombre $2^n + 1$ pour $n < 150$, n'a de diviseur inférieur à 4 600 000, il en résultait que N n'a que 2 facteurs premiers.
(ii) que l'un des diviseurs est $536n + 1$, l'autre $536n + 403$.
(iii) enfin [21.XI.48] que

$$2^{67} + 1 = 3 \cdot 7\,327\,657 \cdot 6\,713\,103\,182\,899."$$

This completes the factorization of $2^n + 1$ up to $n = 70$. Mr. Ferrier is the author of the work on prime numbers which we reviewed *MTAC*, v. 3, p. 95; see also v. 2, p. 341.

101. NEWMAN'S *Mathematical Tracts*.—When we wrote our Note about FRANCIS WILLIAM NEWMAN (1805–1897), and mathematical tables which he had computed and published (*MTAC*, v. 1, p. 454–459), we knew of only the first edition of his *Mathematical Tracts*, Part I, 1888, ii, 1–80 p. and Part II, 1889, iv, 81–139 p. Through information furnished to us by Dr. ALAN FLETCHER we learned that Bowes & Bowes had in 1912 published a reprint of these two parts of the *Tracts*, in a single volume, now out of print. Brown University has recently acquired the last copy in stock.

R. C. A.

102.—TABLES OF x tan x.—Mr. JOHN TODD of King's College, London, has reminded us that we omitted to refer to ENGLUND's table (*MTAC*, v. 2, p. 20) in our EDITORIAL NOTE, *MTAC*, v. 3, p. 296.

QUERIES

30.—GIRARD AND SNELL TABLES.—D. BIERENS DE HAAN, *Bibliographie Néerlandaise Historique-Scientifique des Ouvrages Importants . . . sur les Sciences Mathématiques et Physiques*, Rome, 1883, lists two mathematical tables by these authors. The first published work of Albert Girard (1595–1632), editor of the works of SIMON STEVIN, was *Tables des Sinvs, Tangentes & Secantes, selon le raid de 100000 parties. Avec un traicté succinct de Trigonometrie. . . .* The Hague, Elzevir, 1626, 120 p., of which there is a copy in Library of Congress. Second editions corrected and enlarged (132 p.) in French and Latin were also published by Elzevir in 1629. There is a copy of this French edition in the New York Public Library. The last published book of Willebrord Snell (1580 or 1581–1626), before his death, was *Canon Triangulorum, hoc est sinuum, tangentium et secantium Tabulae, ad taxationem*

radij 100000,00. Leiden, 1626, 181 p. Of this work there is a copy in the Columbia University Library. Where may other copies of these works be inspected? Exactly what is given in the tables? Are they in any way indebted to those of Pitiscus in 1600 and 1608 (or 1612)?

R. C. A.

QUERIES—REPLIES

39. Some Clothoid or Euler Spiral Tables (Q **26**, v. 3, p. 146).—Since this query was published Brown University has acquired a copy of ***Klothoiden-Absteckiafeln*** by Walther Schürba of Brünn, Czechoslovakia, published in Berlin, by Volk und Reich Verlag, 1942, 143 p. 16.5 × 23 cm. L. J. C. has informed us that there is also a copy of this volume in his library. In the preface Schürba states that he was led to prepare his work by becoming acquainted with Prof. Dr. Leopold Örley, ***Übergangsbogen bei Strassenkrümmungen***, Berlin, 1937, where "the first useful practical laying out of the clothoid was recommended." Since Lehmer published his article in 1904, 33 years before Örley, Schürba's "first" is highly erroneous. The use of the clothoid with some tables is indicated in Joseph Barnett, *Transition Curves for Highways*, Washington, 1938. More elaborate discussion occurs in T. F. Hickerson, *Highway Surveying and Planning*, New York, 1936, p. 156–183, etc. So also in Arthur N. Talbot, *The Railway Transition Spiral*, sixth ed., New York, 1927, except that he used the chord definition and his tables were developed on this basis.

R. C. A.

CORRIGENDA

V. 1, p. 198, l. 12–13, and v. 3, p. 268, l. 12, *delete*: K(86°48′), *for* 4.2744, *read* 4.2746; also all references in this connection to errors in Jahnke & Emde, *Tables of Functions*, 1909, 1933, 1938, 1943. P. 273, l. − 8, *for* $(k-1)b = s$, *read* $(k-1)b = \pi s$.

V. 2, p. 230, l. 23, *for* Dr. K. G. Macleish, *read* Mr. R. D. O'Neal; p. 398, col. 2, l. 17, *delete* 230,; p. 400, col. 1, l. 34, *add* 230,.

V. 3, p. 358, l. 11, *for* $\binom{p}{j}$, *read* $\binom{q}{j}$; p. 376, l. − 8, *for* machines, can, *read* machines can; l. − 7, *for* machines and, *read* machines, and; p. 383, *for* Nichola Begonich, *read* Nicholas Begonich; p. 392, l. 5 and 15, *for* ***Bartholomevv***, *read* ***Bartholmevv***; l. 10, *for* Fawcit, *read* Fawcet; l. 11, *for* Favvcit, *read* Favvcet; p. 396, l. 11, *for* Lincoln ***Cathedral***, *read* Lincoln Cathedral; l. 26, *for* cse, *read* csc; l. − 17, *for* fables, *read* tables.

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